BUCKLING AND POSTBUCKLING OF CIRCULAR PLATES CONTAINING CONCENTRIC PENNY-SHAPED DELAMINATIONS

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Abstract—Buckling and postbuckling analyses of circular laminated composite plates with delaminations are presented. An axisymmetric finite element model based on a layer-wise laminated composite plate theory is developed to formulate the problem. Geometric nonlinearity in the sense of von Kármán and imperfections in the form of initial global deflection and initial delamination openings are included. A simple contact algorithm which precludes the physically inadmissible overlapping between delaminated surfaces is proposed and incorporated into the analysis.

Numerical results are obtained addressing the effects of the initial imperfections, the number of delaminations and their sizes on the critical buckling load and buckling mode shapes as well as postbuckling responses.

INTRODUCTION

Axisymmetric models have often been used in the literature for the analysis of circular plates with a concentric penny-shaped delamination. Based on the axisymmetry assumption, such plates loaded by radial compressive loads distributed uniformly around the circumference can be treated as one-dimensional problems without the need for solution of formidable and complicated two-dimensional counterparts.

Axisymmetric buckling and growth of a thin circular delamination in a plate was first studied by Kachanov [1] on the basis of a linear postbuckling solution. Bottega and Maewal [2] developed an analytical model based on asymptotic analysis of postbuckling behavior for a symmetric two-layer isotropic circular plate. It was found that delamination growth based on a Griffith-type fracture criterion may occur following delamination buckling provided that sufficient bending energy is produced at the delamination crack front. In a subsequent paper [3], Bottega and Maewal extended the analysis in [2] to include the dynamics of the delamination growth process resulting from buckling of the delamination.

Critical buckling load of an isotropic circular plate with concentric delamination for various delamination geometries under uniform circumferential loading was investigated by Yin and Fei [4]. Later, Yin [5] investigated postbuckling behavior for an isotropic circular plate containing thin-film or midplane delamination employing von Kármán's nonlinear plate equations. In a subsequent paper [6], Yin and Fei also improved upon Yin's earlier work [5] by including the deformation of the base plate for postbuckling analysis of a delaminated circular plate.

For more complex geometry, numerical procedures such as finite element method were employed. A two-dimensional axisymmetric finite element model was used by Partridge et al. [7] for delaminated isotropic circular plates to obtain strain energy release rate. Bruno and Grimaldi [8] studied both through-the-width and circular delaminations for symmetric two-layer plates. They developed analytical and finite element models based on the von Kármán thin plate theory in conjunction with the unilateral contact approach in which delamination is modeled by means of elastic foundation.

A model which includes a hole at the center of a circular plate and polar orthotropy using nonlinear thin plate theory was proposed by Larsson [9]. He investigated the geometric threshold where the thin film approximation is applicable. Larsson [10] also investigated the effects of multiple delaminations for various combinations of geometry and material parameters. It was reported that the contact effect between delaminated layers is no longer negligible for some cases of multiple delaminations.

In summary, most previous literature assumed that the material is isotropic and one delamination was assumed to exist near the surface or midplane of a plate. Furthermore, it appears that postbuckling response of laminates with delaminations was limited to perfect plates, and imperfection sensitivity was ignored. In earlier studies, a layer-wise laminate analysis based finite element model was developed for buckling and postbuckling of delaminated composites by the present authors [11, 12], and applied to beams with through-the-width delaminations. In the present paper, buckling and postbuckling of cylindrically orthotropic circular plates containing single and multiple delaminations are considered. Imperfections
of the plate in the form of initial deflection and initial delamination openings are included, and the contact algorithm developed in [12] is used to preclude the physically inadmissible overlapping between delaminated surfaces.

THEORETICAL FORMULATION

Kinematics

An N-layer fiber-reinforced circular composite plate containing multiple delaminations is considered here. A cylindrical coordinate system (r, θ, z) is introduced in such a way that the center of the plate coincides with the origin of the coordinate system (Fig. 1). As a result of axisymmetry, the displacements are independent of θ coordinate. As in the layer-wise theory in the Cartesian coordinates previously developed by the present authors [12], the assumed displacement field is supplemented with unit step functions to model delaminations. Thus, the resulting radial and out-of-plane displacements $U_r$ and $U_z$ at a generic point $r, z$ in the laminate are assumed to be of the following form due to the axisymmetric assumption (Fig. 2):

$$U_r(r, z) = u_r(r) + \Phi(z)u_{r1}(r) + \delta(z)\Phi''(r),$$  
$$U_z(r, z) = u_z(r) + \Phi(z)u_{z1}(r),$$  

where $i$ and $j$ indices range from 1 to $D$ and 1 to $N$, respectively, where $D$ is the number of delaminations and $N$ is the number of layers. Repeated superscripts denote summation convention.

The terms $u_r$ and $u_z$ are the displacements of a point $(r, 0)$ on the reference surface of the laminate, $u_{r1}$ and $u_{z1}$ are nodal values of the displacements in the r direction of each layer, and $\Phi$ and $\Phi''$ represent the relative slipping and opening displacements, respectively, across the $L(i)$th delaminated interface where $L(i)$th denotes the location of the interface where the $i$th delamination lies (Fig. 2). $\Phi(z)$ and $\Phi''(z)$ are the linear Lagrangian interpolation function through the thickness of the laminate, and the unit step function, respectively.

Following the procedure in Chia [13], the nonlinear strain tensor with initial plate curvature in the sense of von Kármán nonlinearity is derived. Neglecting the quadratic terms involved in the radial displacements $u_r$, $u_z$ and $\Phi''$, the measure of strain is given by

$$ds^2 - ds_0^2 = 2(\epsilon_r r^2 + \epsilon_\theta r^2 d\theta^2 + \gamma_{r\theta} dr dz),$$  

where $\epsilon_r$, $\epsilon_\theta$ and $\gamma_{r\theta}$ are the nonlinear strains given by

$$\epsilon_r = \epsilon_r^0 + \Phi'\epsilon_r^i + \delta\epsilon_r^p,$$  
$$\epsilon_\theta = \epsilon_\theta^0 + \Phi'\epsilon_\theta^i + \delta\epsilon_\theta^p,$$  
$$\gamma_{r\theta} = \gamma_{r\theta}^0 + \Phi'\gamma_{r\theta}^i + \delta\gamma_{r\theta}^p,$$  

and

$$\epsilon_{r1} = u_{r1}, \quad \epsilon_{z1} = u_{z1} + \delta u_{z1}.$$  

In eqns (7)-(10), $\xi$ and $\xi^0$ denote globally deflected shape and shape of an initial opening of each delamination, respectively.

Variational formulation

In order to derive the governing equilibrium equations of the present theory, the principle of virtual displacements is used. In the absence of body forces, we have

$$0 = \int_V \left( \sigma_r \delta \epsilon_r + \sigma_\theta \delta \epsilon_\theta + \sigma_{r\theta} \delta \gamma_{r\theta} - f_r \delta U_r - f_\theta \delta U_\theta \right) dV - \int_S \left( \vec{t}_r \delta U_r + \vec{t}_\theta \delta U_\theta \right) dS,$$  

Fig. 1. Geometry of an axisymmetric problem with delaminations.
where \( \hat{t}_r \) and \( \hat{t}_z \) are the specified radial and vertical components of surface traction forces, and \( f_r \) and \( f_z \) are the body forces acting in the radial and vertical directions, respectively. In eqn (13), \( V \) is the volume of the plate and \( S \) is the boundary of \( V \) on which the tractions are specified.

After substituting eqns (4), (5) and (6), we perform integration through-the-thickness of eqn (13) to produce

\[
0 = 2\pi \int_0^R \left[ N_r \frac{\partial u_r}{\partial r} + \frac{\partial u_r}{\partial z} (u_z + \xi_v) + \frac{1}{r} N_\theta \frac{\partial u_z}{\partial \theta} + Q_r \frac{\partial u_r}{\partial \theta} + \frac{1}{r} N_\theta \frac{\partial u_z}{\partial \theta} + Q_\theta \frac{\partial u_z}{\partial \theta} + N_r \frac{\partial u_r}{\partial \theta} + \frac{1}{r} N_\theta \frac{\partial u_z}{\partial \theta} \right] \, dr
\]

and the specified traction resultants are defined by

\[
(N_r, N_\theta, N_r') = \int_{-h/2}^{h/2} \sigma_r [1, \phi', \delta', \delta''] \, dz \quad (15)
\]

\[
(N_\theta, N_r, N_\theta') = \int_{-h/2}^{h/2} \sigma_\theta [1, \phi', \delta', \delta''] \, dz \quad (16)
\]

\[
(Q_r, Q_\theta, Q_{r'}) = \int_{-h/2}^{h/2} \sigma_{r'} \left[ \frac{1}{r} \frac{1}{r}, \phi', \delta' \right] \, dz \quad (17)
\]

**Equilibrium equations**

The equilibrium equations of the present theory can be derived by integrating the derivatives of the varied quantities by parts and collecting the coefficients of \( \delta u_r \), \( \delta u_z \), \( \delta u_r' \), \( \delta u_z' \), \( \delta u_r'' \), \( \delta u_z'' \),

\[
\frac{1}{r} \left( r N_r \right)_r - N_\theta = 0 \quad (20)
\]
where \([Q]^{(k)}\) denotes the transformed reduced stiffness matrix of the kth layer.

Substitution of eqn (32) into eqns (15)-(17) gives the constitutive equations of the laminate:

\[
\begin{bmatrix}
\frac{1}{r} (rN_r)_{,r} - \frac{N_0}{r} - Q_r = 0 \\
\frac{1}{r} (rN_r)_{,r} - \frac{N_0}{r} = 0 \\
\frac{1}{r} (rN_r)_{,r} - \frac{N_0}{r} = 0
\end{bmatrix}
\]

The equilibrium equations consist of \((2 + N + 2D)\) differential equations in \((2 + N + 2D)\) variables \((u, u', u'', \ldots)\). Buckling equations can be derived by neglecting initial imperfection functions and linearizing the equilibrium equations.

The boundary conditions are of the form

\[
N_r - \tilde{N}_r = 0 \quad \text{or} \quad u_r = \tilde{u}_r \\
Q_r - \tilde{Q}_r = 0 \quad \text{or} \quad u_r = \tilde{u}_r \\
N'_r - \tilde{N}'_r = 0 \quad \text{or} \quad u'_r = \tilde{u}'_r \\
\tilde{Q}_r = 0 \quad \text{or} \quad \tilde{u}_r = \tilde{\tilde{u}}_r
\]

where

\[
Q_r = N_r (u + \xi) \times \tilde{N}_r (\tilde{u} + \tilde{\xi})_{,r} + Q_r \\
\tilde{Q}_r = N_r (u + \xi) \times \tilde{N}_r (\tilde{u} + \tilde{\xi})_{,r} + \tilde{Q}_r
\]

Constitutive equations

The constitutive equations of a kth orthotropic lamina in the laminate co-ordinate system are given by

\[
\begin{bmatrix}
\sigma_r \\
\sigma_{r'} \\
\sigma_{r''}
\end{bmatrix}^{(k)} = \begin{bmatrix}
\tilde{Q}_{11} & \tilde{Q}_{12} & 0 \gamma_r \\
\tilde{Q}_{12} & \tilde{Q}_{22} & 0 \gamma_r \\
0 & 0 & \tilde{Q}_{33}
\end{bmatrix} \begin{bmatrix}
\epsilon_r \\
\epsilon_{r'} \\
\epsilon_{r''}
\end{bmatrix}^{(k)}
\]

where

\[
(A_{55}, B_{55}, E_{55}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{55}^{(k)} (1, \phi_r, \phi') \, dz
\]

\[
(D_{55}^{(k)}, F_{55}^{(k)}, E_{55}^{(k)}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{55}^{(k)} (1, \phi_r, \phi') \, dz
\]

\[
(F_{55}^{(k)}, E_{55}^{(k)}, E_{55}^{(k)}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{55}^{(k)} (1, \phi_r, \phi') \, dz
\]

Finite element model

The generalized displacements \((u, u', u'', \ldots)\) are expressed over each element as a linear combination of the one-dimensional Lagrangian interpolation function \(\psi_i\) associated with node \(l\) and the total values \((u), (u'), (u''), \ldots)\) at

\[
u_l = \sum_{i=1}^{n} (u_i) \psi_i
\]

\[
u_l = \sum_{i=1}^{n} (u_i) \psi_i
\]
where $n$ is the number of nodes in a typical finite element.

Substituting these expressions into the weak statement in eqn (14), the finite element model of a typical element for postbuckling analysis can be obtained as

\[ [K(\Delta)] \{\Delta\} = \{F\}, \quad (41) \]

where $[K(\Delta)]$ is the element stiffness matrix given by

\[ [K] = \begin{bmatrix}
\end{bmatrix}_{\text{Symm.}} + \begin{bmatrix}
[0] & [0] & [0] & [0] & [1K] \\
[0] & [0] & [0] & [0] & [1K] \\
\end{bmatrix}_{NL} \quad (42) \]

Note that the linear and nonlinear parts of the stiffness matrix are distinguished by using subscripts $N$ and $NL$, respectively, and the nonlinear part $[K]_{NL}$ is not symmetric. $\{F\}$ is the element force vector containing boundary and transverse force terms

\[ \{F\} = \{1F\} \{2F\} \{3F\} \{4F\} \{5F\}^T. \quad (43) \]

The element stiffness matrix $[K]$ and the force vector $\{F\}$ are presented in the Appendix. $\{\Delta\}$ is the nodal displacement vector given by

\[ \{\Delta\} = \{u_1\} \{u_2\} \{u_3\} \{u_4\} \{u_5\}^T. \quad (44) \]

It is possible that the solution of eqn (41) may indicate overlapping of the delaminated sublaminates during the postbuckling deformation depending on the direction and magnitude of initial imperfection [12]. Thus, for accurate postbuckling analysis of delaminated composites, physically inadmissible sublaminate overlapping phenomena must be prevented. In the present study, the contact algorithm developed in [12] is employed.

For buckling analysis, the finite element model of a typical element can be obtained as the eigenvalue problem

\[ ([K]_L - \lambda [G]) \{\Delta\} = \{0\}, \quad (45) \]

where $[K]_L$ is the linear part of the element stiffness matrix and $[G]$ is the element geometric stiffness matrix given by

\[ [G] = \begin{bmatrix}
[0] & [0] & [0] & [0] & [0] \\
[0] & [0] & [0] & [0] & [0] \\
[0] & [0] & [0] & [0] & [0] \\
\end{bmatrix}_{\text{Symm.}} \quad (46) \]

In eqn (45), $\lambda$ refers to a buckling coefficient and $\{\Delta\}$ is the buckling mode.

**NUMERICAL RESULTS AND DISCUSSION**

Because of its axisymmetry, the problem can be simplified to a one-dimensional problem and an axisymmetric finite element can be used. Although the present formulation can be applied to polar anisotropic materials, all the examples considered for illustrative purposes are isotropic materials.

**Delamination buckling**

A clamped isotropic circular plate containing a concentric penny-shaped delamination at its midplane is considered (Fig. 3). The plate is under uniform circumferential compressive load. For convenience, the following non-dimensional quantities are defined:

\[ \tilde{a} = a/R, \quad \tilde{t} = t/h, \quad (47) \]

where $t$ is the thickness of a delaminated upper layer, $h$ is the total thickness of the plate, $a$ is the radius of the delamination, and $R$ is the radius of the plate. The critical buckling loads with respect to the delamination radius are presented for two different thickness-to-radius ratios. In the first case, the thickness-to-radius ratio is assumed to be very small ($h/R = 1/500$), so that the results can be compared with the previous results of classical lamination theory [9]. Secondly, a moderately thick plate ($h/R = 1/10$) is considered to order to investigate the...
effect of thickness-to-radius ratio on the buckling loads and buckling modes.

Three distinguishable buckling mode shapes obtained based on linear bifurcation analysis are shown in Fig. 4. The mode shapes have \( m \) diametrical nodes and \( n \) circular nodes. The buckling coefficients for the three distinguishable buckling modes are shown in Figs 5 and 6. In these figures, the buckling loads are normalized as

\[
P = \frac{NR^2}{D_t},
\]

where

\[
D_t = \frac{Eh^3}{12(1-\nu^2)},
\]

and \( N \) is the in-plane compressive load applied to the plate. The present results show excellent agreement with the results reported by Larsson [9] for a model with \( m = 0, n = 1 \) as shown in Fig. 5. It is noted that, unlike the case of through-the-width delaminations [11], the slenderness of the plate hardly affects buckling loads and modes. For all of the range of delamination radius, the mode with \( m = 0, n = 1 \) is dominant, and the buckling load for the mode with \( m = n = 1 \) is far beyond that of the mode with \( m = 0, n = 1 \). The buckling load for local mode is substantially reduced with respect to the delamination size, but does not get smaller than the buckling load of \( m = 0, n = 1 \).

**Delamination postbuckling**

As a verification, the present postbuckling solutions are compared with the previous results [6] in Table 1 for the case of near-the-surface delamination (\( \delta = 0.1 \)). In Table 1, \( P \) and \( U \) are non-dimensional

Fig. 3. Geometry of a circular plate containing a concentric penny-shaped delamination.

Fig. 4. Three distinguishable buckling modes for a circular plate having a penny-shaped delamination at midplane.
Fig. 5. Buckling loads vs delamination radius for a circular plate containing a single midplane penny-shaped delamination with $h/R = 1/500$.

quantities of applied compressive load and radial shortening at the edges, respectively, defined as

$$P = \frac{N}{N_c}, \quad U = \{12(1 - \nu^2)u_r(R)/R\}_{ij} = \begin{pmatrix} a \\ \bar{a} \end{pmatrix}, \quad (50)$$

where $N_c$ is the buckling load of the undelaminated plate and $u_r(R)$ is a radial displacement at the supports. For two different delamination sizes, $\bar{a} = 0.125$ and $0.25$, radial shortenings of the plate are presented with respect to the applied load. The present results show good agreement with those of Yin and Fei [6].

As another example, postbuckling solutions are obtained for clamped circular plates with $\bar{a} = 0.3$ and $\bar{a} = 0.2$. The initial imperfect function is taken as

$$\zeta = \frac{1}{2}w_0 h \left(1 - \cos \frac{\pi r}{R}\right), \quad (51)$$

where $w_0$ is the fraction of initial deflection with respect to the plate thickness at the center. Initial

Table 1. Comparison of postbuckling solutions for a penny-shaped delamination at midplane of a circular plate

<table>
<thead>
<tr>
<th>$a/R$</th>
<th>$U$</th>
<th>$P$</th>
<th>Yin and Fei [6]</th>
<th>Present</th>
<th>$U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125</td>
<td>3.2098</td>
<td>3.2100</td>
<td>0.6416</td>
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</tr>
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<td>3.2219</td>
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</tr>
<tr>
<td>3.2436</td>
<td>3.2436</td>
<td>0.6515</td>
<td>0.1638</td>
<td>3.2436</td>
<td>3.2441</td>
</tr>
<tr>
<td>3.2943</td>
<td>3.2945</td>
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<td>0.1689</td>
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<td>3.2942</td>
</tr>
<tr>
<td>3.3432</td>
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<td>0.1740</td>
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</tr>
<tr>
<td>3.4384</td>
<td>3.4384</td>
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<tr>
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<tr>
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<td>0.8857</td>
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<td>7.5748</td>
</tr>
</tbody>
</table>
delamination openings $w_0$ are assumed to be zero for all the cases considered.

The load–deflection curves with initial imperfections of various magnitudes are shown in Fig. 7. The bifurcation analysis predicts that the nondimensional buckling load is $P = 0.435$, which is indicated by the horizontal line in Fig. 7. The out-of-plane deflection is normalized as $W = w_c/h$ at the center of the plate. For all the cases considered, the contact algorithm is applied, and thus the load–deflection curves do not show any overlapping. For small positive initial imperfection ($w_0 = 0.001$), the thin top layer starts to deflect upward while the bottom layer does not deflect at delamination buckling load. At $P = 0.9$, the deformation of the bottom layer is accelerated, rendering the top layer substantially deflected downward. The deformation pattern of the plate is significantly different for the case of $w_0 = 0.1$. That is, the delamination buckling load is not distinguishable at all. The plate starts to deflect upward with a small opening of the delamination as soon as the load is applied, and the two layers contact each other above the load level of $P = 0.8$. For $w_0 = -0.001$, the load–deflection curve shows almost the same path as for the case of the same magnitude of imperfection with positive direction. This implies that when the magnitude of initial imperfection is sufficiently small compared to the plate thickness, the plate follows the same equilibrium path regardless of the direction of the imperfection as noticed for the through-the-width delamination case [12]. The load–deflection curve for $w_0 = -0.1$ shows that the delamination buckling occurs at a slightly higher load level than for the bifurcation analysis. Until the buckling occurs, both surfaces stay together with a relatively small contact force. At the buckling load, this contact force is suddenly increased due to the top layer, which tries to buckle down. Since the bending stiffness of the bottom layer is much higher than that of the top layer, the bottom layer does not deflect, and the top layer snaps through and buckles up. Accordingly, the plate deflects down with a large delamination opening.

Load vs radial-shortening curves are shown in Fig. 8 for plates with $\bar{r} = 0.2$ and different sizes of delaminations ($\alpha = 0.3, 0.6$ and 0.9). Buckling loads for the other two cases ($\bar{a} = 0.6$ and 0.9) are found to be $P = 0.110$ and $P = 0.049$, respectively. All the three buckling loads are denoted by the three horizontal lines in Fig. 8 for each delamination length. For all the cases, $w_0$ is set to 0.001. In general, as delamination size increases, the load-carrying capacity of the plate decreases as can be expected. For a relatively small delamination ($\alpha = 0.3$), the curve hardly changes its slope at the buckling load, and then a gradual reduction of load-carrying capacity starts to occur near $P = 0.9$. Consequently, the load carrying capacity does not show significant difference from the undelaminated plate in this case. For the case of $\alpha = 0.6$, the curve slightly changes its slope near the buckling load. However, the subsequent reduction in load-carrying capacity does not occur until the load reaches $P = 0.7$, where the load-carrying capacity is gradually reduced. For a large delam-
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Fig. 9. Load-deflection curve for a circular plate containing three equidistant penny-shaped delaminations with \( a_1 = 0.4, a_2 = 0.3, a_3 = 0.2 \).

The effects of contact behavior of delaminated surfaces and initial imperfections of composites on postbuckling behavior are also investigated.

Based on the above theoretical developments and numerical results, the following concluding remarks are made:

1. The thickness-to-span ratio of a delaminated composite does not cause substantial change in buckling loads and mode shapes for penny-shaped midplane delaminations in circular plates under circumferential load. The buckling modes are always \( m = 0, n = 1 \).
2. The postbuckling response of a delaminated composite is very sensitive to the amplitude and the
direction of the initial imperfection. This is because there are many equilibrium paths, and the plate can take any path depending on the initial imperfection.

(3) For composites containing multiple delaminations, the contact effects between delaminated surfaces become significant, and the proposed simple contact algorithm is shown to be very efficient for solving the problem.

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REFERENCES


APPENDIX

The linear components of the stiffness matrix in eqn (42) are given by

\[ (^{(1)}K_{ij})_L = \int \left[ rB_{1i} \psi_i \psi_j + B_{1i}' \left( \psi_i \psi_j + \psi_i' \psi_j' \right) \right] dr + \frac{1}{r} B_{2i} \psi_i' \psi_j' \]

\[ (^{(4)}K_{ij})_L = \int \left[ rE_{1i} \psi_i \psi_j + E_{1i}' \left( \psi_i \psi_j + \psi_i' \psi_j' \right) \right] dr + \frac{1}{r} E_{2i} \psi_i' \psi_j' \]

\[ (^{(3)}K_{ij})_L = \int \left[ rD_{1i} \psi_i \psi_j + D_{1i}' \left( \psi_i \psi_j + \psi_i' \psi_j' \right) \right] dr + \frac{1}{r} D_{2i} \psi_i' \psi_j' \]

\[ (^{(2)}K_{ij})_L = \int \left[ r \left( \psi_i \psi_j + \psi_i' \psi_j' \right) \right] dr + \frac{1}{r} \psi_i' \psi_j' \]

\[ (^{(5)}K_{ij})_L = \int \left[ r \left( \psi_i \psi_j + \psi_i' \psi_j' \right) \right] dr + \frac{1}{r} \psi_i' \psi_j' \]

\[ (^{(6)}K_{ij})_L = \int \left[ \psi_i \psi_j + \psi_i' \psi_j' \right] dr \]
The nonlinear components of the stiffness matrix in eqn (42) are given by

\[ (12K_{ij})_{NL} = \int \left[ \frac{1}{2} r (A_{11} u_{r,r} + E_{11} \bar{u}_{r,r}) \psi_r \psi_r \right] dr \\
+ (A_{12} u_{r} + E_{12} \bar{u}_{r}) \psi_r \psi_r \] \\

\[ (15K_{ij})_{NL} = \int \left[ \frac{1}{2} r (E_{11} u_{r} + E_{11} \bar{u}_{r}) \psi_r \psi_r \right] dr \\
+ (E_{12} u_{r} + E_{12} \bar{u}_{r}) \psi_r \psi_r \] \\

\[ (22K_{ij})_{NL} = \int \left[ r (A_{11} u_{r,r} + 3 \bar{u}_{r,r}) \right] dr \\
+ E_{11} u_{r,r} \bar{u}_{r,r} + 3 \bar{u}_{r,r} + E_{11} \bar{u}_{r,r} (u_{r,r} + 3 \bar{u}_{r,r}) \\
+ E_{11} \bar{u}_{r,r} \bar{u}_{r,r}(u_{r,r} + 3 \bar{u}_{r,r}) \] \\

The tangent stiffness matrix is given as

\[ (12K_{ij})_T = (12K_{ij})_L + (12K_{ij})_{NL} \\
+ \int \left[ N_{1,1} u_{r} + \frac{1}{2} (A_{11} u_{r,r} + E_{11} \bar{u}_{r,r}) (u_{r,r} + \bar{u}_{r,r}) \right] dr \\
+ \int \left[ (E_{11} u_{r} + E_{11} \bar{u}_{r}) (u_{r} + \bar{u}_{r}) \right] dr \\
]

\[ (15K_{ij})_T = (15K_{ij})_L + (15K_{ij})_{NL} \\
+ \int \left[ N_{1,1} u_{r} + \frac{1}{2} (E_{11} u_{r} + E_{11} \bar{u}_{r}) (u_{r} + \bar{u}_{r}) \right] dr \\
+ \int \left[ (E_{11} u_{r} + E_{11} \bar{u}_{r}) (u_{r} + \bar{u}_{r}) \right] dr \\
]

\[ (22K_{ij})_T = \frac{1}{2} (22K_{ij})_L + (22K_{ij})_{NL} \\
+ \int \left[ N_{1,1} u_{r} + \frac{1}{2} (E_{11} u_{r} + E_{11} \bar{u}_{r}) (u_{r} + \bar{u}_{r}) \right] dr \\
+ \int \left[ (E_{11} u_{r} + E_{11} \bar{u}_{r}) (u_{r} + \bar{u}_{r}) \right] dr \\
]

All the other components of tangent stiffness matrix are given as

\[ [K]_T = [K]_L + [K]_{NL}. \]

The force vector in eqn (43) is

\[ F_i = N_i \psi_r \] \\
\[ F_i = Q_i \psi_r + \int p \psi_r dr \] \\
\[ F_i = N_i \psi_r \] \\
\[ F_i = Q_i \psi_r \] \\
\[ F_i = Q_i \psi_r \]

The components of the geometry stiffness matrix in eqn (46) are given by

\[ G_{ij} = \int n \psi_{i, r} \psi_{j, r} dr \]

\[ G_{ij} = \int \psi_{i, r} \psi_{j, r} dr \]

\[ G_{ij} = \int n \psi_{i, r} \psi_{j, r} dr \]

\[ G_{ij} = \int n \psi_{i, r} \psi_{j, r} dr \]