THERMALLY INDUCED BUCKLING OF LAMINATED COMPOSITES BY A LAYERWISE THEORY

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Abstract—Thermal buckling analysis of laminated composites is presented by using a layer-wise theory. Governing buckling equations are derived from the variational principle and a finite element method is developed to formulate the problem. Numerical results are obtained and compared with those of other theories addressing the effects of the thickness-to-span ratio, lamination angle, the ratio of thermal expansion coefficients and degree of orthotropy on buckling temperature for antisymmetric angle-ply laminates. It is found that a layer-wise approach may be necessary for more accurate thermal buckling analysis of laminated composites. © 1997 Elsevier Science Ltd.

I. INTRODUCTION

Fiber-reinforced composite materials have been increasingly used over the past few decades in aerospace structures that require a high ratio of stiffness and strength to weight. These structures are often subjected to severe thermal environments during launching and re-entry, and thermal load becomes a primary design factor in certain cases. Thermal buckling of laminated composites is, therefore, a subject of vital importance in view of the widespread use of laminated composites in aerospace applications.

One of the earliest studies to examine thermal buckling of plates is that conducted by Gossard et al. [1]. They used the Rayleigh-Ritz method to calculate buckling temperature of simply-supported isotropic rectangular plates. Many other investigations have been done for thermal buckling of isotropic [2-5] or orthotropic plates [6].

It appears that the thermal buckling problem of laminated composite plates has also been studied by several researchers. Chen and Chen [7] used the Galerkin method to study thermal buckling of antisymmetric angle-ply laminates based on the classical lamination theory (CLT). The buckling temperatures for various lamination parameters and boundary conditions under uniformly distributed temperature were presented. They also extended their work to include a nonuniform temperature field by using the finite element method [8]. Thangaratnam et al. [9] also used the finite element method to study the thermal buckling of symmetric and antisymmetric cross-ply and angle-ply laminated plates. The effects of aspect ratio, moduli ratio, temperature distribution, thermal expansion coefficient ratio and number of layers on the buckling temperature were examined.

It is well known that the classical lamination theory, which neglects transverse shear deformation, is inadequate for the analysis of laminated composites, and the shear deformable theory should be introduced. In this regard, Tauchert [10] studied the thermal buckling behavior of antisymmetric angle-ply laminates based on the first order shear deformable plate theory (FSDT). Mathew et al. [11] used the finite element method to calculate the buckling temperature of symmetric and asymmetric cross-ply composite beams based on the first order shear deformable plate theory.

To account for the effect of transverse normal strain on the thermal buckling of laminated composites, a higher order shear deformable plate theory (HSDT) was proposed by Chang and Leu [12] and Chang [13]. According to their work, CLT and FSDT would yield erroneous results regardless of the thickness-to-span ratio of a plate. It was indicated that this discrepancy is mainly because of the transverse normal strain. However, recently, the results reported in Chang and Leu [12] have been found to be invalid [14] due to neglecting the stress-free boundary condition at the top and bottom surfaces of a plate. It is reported that by using the stress-free boundary condition, the discrepancy between FSDT and the theory proposed by Chang and Leu [12] is negligibly small even for a moderately thick plate. Noor and Burton [15] used predictor-corrector procedures for thermal buckling analysis of laminated composite plates, and compared the results with the analytical three-dimensional thermoelasticity solution. It is shown that the predictor-corrector method delivers very accurate results.
All the literature about laminated composites discussed above except Noor and Burton [15] used equivalent single-layer two-dimensional theories (CLT, FSDT, HSDT), which model a multi-layered composite as an equivalent single-layer homogeneous plate. In this equivalent single-layer theory, the coupling effects coming from material anisotropy may be overlooked. A layer-wise theory, which models laminates layer-by-layer, becomes therefore necessary for more accurate thermal buckling analysis of laminated composites. It appears that thermal buckling analyses based on a layer-wise concept have not yet been treated in the open literature to the author’s knowledge.

In the present study, the thermoelastic version of the layer-wise theory [16] is used for the thermal buckling analysis of laminated composite plates subjected to a uniform temperature field. The thermal buckling equations of the layer-wise theory are obtained by the application of the variational principle, and the finite element method is developed to formulate the problem. Numerical results are presented for antisymmetric angle-ply laminates addressing the effect of thickness-to-span ratio, lamination angle, thermal expansion coefficient ratio and the degree of orthotropy on buckling temperature.

II. THEORETICAL FORMULATION

A. Kinematics

An N-layer fiber-reinforced composite plate is considered. The resulting displacements $U_i$ and $U_j$ at a generic point $x_1, x_2, x_3$ in the laminate are assumed to be of the form:

$$U_i (x_1, x_2, x_3, t) = u_i (x_1, t) + \phi (x_1) u_i (x_1, t), \quad (1a)$$

$$U_j (x_1, x_2, x_3, t) = u_j (x_2, t). \quad (1b)$$

The usual Cartesian indicial notation is employed where Greek subscripts are assumed to have values 1–2. Superscript $j$ ranges from 1 to $N$, where $N$ is the number of layers. The terms $u_i$ and $u_j$ are the displacements of a' point $(x_1, t)$ on the reference surface of the laminate, $u_i'$ are nodal values of the displacements in the $x_i$ direction of each single layer, and $\phi(x)$ is the linear Lagrangian interpolation function through the thickness of the laminate which accounts for linear variation of displacement field within each layer.

Neglecting the quadratic terms involved in inplane displacements according to von Kármán assumptions, the nonlinear strain tensors are given by

$$\varepsilon_{ij} = \varepsilon_{ij}' + \phi \varepsilon_{ij}', \quad (2a)$$

$$\gamma_{ij} = \gamma_{ij}' + \phi \gamma_{ij}', \quad (2b)$$

where

$$\varepsilon_{ij}' = \frac{1}{2} (u_{i,x} + u_{j,x} + u_{i,x} u_{j,x}), \quad (3a)$$

$$\varepsilon_{ij}' = \frac{1}{2} (u_{i,x} + u_{j,x}), \quad (3b)$$

$$\gamma_{ij}' = u_{i,x} + \gamma_{ij}'. \quad (3c)$$

B. Constitutive equations

The constitutive equations for thermoelastic bodies are:

$$\varepsilon_{ij} = S_{ij} \sigma_{ij} + \sigma_{ij} \Delta T, \quad (4a)$$

or

$$\sigma_{ij} = C_{ijkl} (\sigma_{kl} - \sigma_{kl} \Delta T). \quad (4b)$$

According to the CLT assumption, the thermoelastic constitutive equations of a $k$th orthotropic lamina in the laminate co-ordinate system become

$$\sigma_{ij}^{(k)} = C_{ijkl}^{(k)} (\varepsilon_{ij}^{(k)} - \varepsilon_{ij}^{(k)} \Delta T), \quad (5)$$

where $C_{ijkl}^{(k)}$ are the transformed reduced stiffnesses, $\varepsilon_{ij}^{(k)}$ are the transformed coefficients of thermal expansion of the $k$th layer, respectively, and $\Delta T$ denotes temperature rise.

C. Variational formulation

The total potential energy of the system can be stated in the absence of body force as

$$\Pi = \frac{1}{2} \int \left[ \sigma_{ij} (\varepsilon_{ij} - \varepsilon_{ij} \Delta T) + \sigma_{ij} \gamma_{ij} \right] dV. \quad (6)$$

Substituting eqns (2) and (3) into eqn (6) and integrating with respect to $x_3$, the total potential energy becomes

$$\Pi(u_i, u_j, u_i) = \frac{1}{2} \int \left[ (N_{ij} - N_{ij} \Delta T) \varepsilon_{ij}^{(k)} + (N_{ij} - N_{ij} \Delta T) \varepsilon_{ij}^{(k)} + Q_{ij} \gamma_{ij}^{(k)} \right] dV. \quad (7)$$

where the stress resultants are defined as

$$[N_{ij}, N_{ij}] = \int_{x_3} \sigma_{ij} [1, \phi] dx_3, \quad (8a)$$

$$[Q_{ij}, Q_{ij}] = \int_{x_3} \sigma_{ij} [1, \phi'] dx_3. \quad (8b)$$
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where

\[ \varepsilon_{ef} = \frac{1}{2} (\varepsilon_{u_{1},\phi} + \varepsilon_{u_{2},\phi}) + \frac{1}{2} (\varepsilon_{u_{3},\phi} + \varepsilon_{u_{4},\phi}) \]

(14a)

\[ \varepsilon_{ef} = \frac{1}{2} (\varepsilon_{u_{1},\phi} + \varepsilon_{u_{2},\phi}) \]

(14b)

\[ \varepsilon_{ef} = \frac{1}{2} (\varepsilon_{u_{3},\phi} + \varepsilon_{u_{4},\phi}) \]

(14c)

\[ \gamma_{\phi} = \gamma_{u_{1},\phi} + \gamma_{u_{2},\phi} \]

(14d)

The stress resultants have the following form:

\[ N_{ef} = N_{u_{1},\phi} + \varepsilon N_{u_{2},\phi} + \varepsilon^{2} N_{u_{3},\phi} \]

(15a)

\[ N_{ef} = N_{u_{2},\phi} + \varepsilon N_{u_{1},\phi} + \varepsilon^{2} N_{u_{3},\phi} \]

(15b)

\[ Q_{ef} = Q_{u_{1},\phi} + \varepsilon Q_{u_{2},\phi} \]

(15c)

\[ Q_{ef} = Q_{u_{2},\phi} + \varepsilon Q_{u_{1},\phi} \]

(15d)

D. Buckling equations

The governing equations of thermal buckling can be derived from the second variation of the total potential energy. In order to obtain the increment in total potential energy, an increment is given to the displacement fields of the equilibrium state.

\[ u_{1} = u_{1} + \varepsilon u_{1} \]

(12a)

\[ u_{2} = u_{2} + \varepsilon u_{2} \]

(12b)

\[ u_{3} = u_{3} + \varepsilon u_{3} \]

(12c)

where \( u_{1}, u_{2}, u_{3} \) and \( u_{4} \) are displacements at the onset of buckling and \( \varepsilon u_{1}, \varepsilon u_{2}, \varepsilon u_{3}, \) and \( \varepsilon u_{4} \) are displacements after the onset of buckling. By substituting eqn (12) into eqn (2), the resulting strains have the following form:

\[ \varepsilon_{ef} = (\varepsilon_{u_{1},\phi} + \varepsilon\varepsilon_{u_{2},\phi} + \varepsilon^{2}\varepsilon_{u_{3},\phi}) \]

(13a)

\[ \gamma_{\phi} = (\gamma_{u_{1},\phi} + \gamma\gamma_{u_{2},\phi}) \]

(13b)

Governing equations for buckling and thermal buckling analysis can be derived by integrating the
derivatives of the varied quantities by parts and collecting the coefficients of \( \delta u_{i1} \), \( \delta u_{i2} \), \( \delta u'_{i} \):

\[
N_{s}, = 0
\]  
(19a)

\[
Q_{i1,i2} + N_{s} \delta u_{i2} = 0
\]  
(19b)

\[
N_{s} \delta u'_{i} = Q_{i2} = 0
\]  
(19c)

The boundary conditions are of the form:

\[
\text{essential natural}
\]

\[
\begin{align*}
\delta u_{i1} & = 0 \\
\delta u_{i2} & = 0 \\
\delta u'_{i} & = 0
\end{align*}
\]
(20a)

\[
\begin{align*}
\delta u_{i1} & = 0 \\
\delta u_{i2} & = 0 \\
\delta u'_{i} & = 0
\end{align*}
\]
(20b)

\[
\begin{align*}
\delta u_{i1} & = 0 \\
\delta u_{i2} & = 0 \\
\delta u'_{i} & = 0
\end{align*}
\]
(20c)

where \( n_{f} \) is a unit normal vector in \( \beta \) direction.

For uniform temperature rise in thermal buckling problem, \( N_{s} \) in eqn (19b) is given by

\[
N_{s} = -\nabla T n_{f} = -\Delta T n_{f},
\]
(21)

where \( \Delta T \) is buckling temperature, and \( n_{f} \) are thermal stress resultants per unit change in temperature and can be written as:

\[
n_{f} = \sum_{k=1}^{N} \int_{-l}^{l} \frac{\partial \phi_{i} (y)}{\partial y} d\Omega_{i}.
\]
(22)

In what follows, subscript 1 in displacements will be dropped for convenience.

E. Finite element model

The generalized displacements \( (u_{i1}, u_{i2}, u'_{i}) \) are expressed over each element as a linear combination of the Lagrangian interpolation function \( \psi_{i} \) associated with node \( l \) and the nodal values \( (u_{i1}), (u_{i2}), (u'_{i}) \):

\[
\begin{align*}
\delta u_{i} & = \sum_{i=1}^{n} (u_{i}); \psi_{i}, \\
\delta u_{i2} & = \sum_{i=1}^{n} (u_{i2}); \psi_{i}, \\
\delta u'_{i} & = \sum_{i=1}^{n} (u'_{i}); \psi_{i},
\end{align*}
\]
(23a)

(23b)

(23c)

where \( n \) is the number of nodes in a typical finite element. Substituting these expressions into the weak statement in eqn (18), the finite element model of a typical element can be obtained as

\[
(K - \Delta T G)\delta = 0,
\]
(24)

where \( K \) is the element stiffness matrix,

\[
K = \begin{bmatrix}
\sigma K & 0 & \sigma K \\
0 & -\sigma K & 0 \\
\sigma K & 0 & \sigma K \\
\end{bmatrix}
\]

(25)

and \( G \) is the element geometric stiffness matrix,

\[
G = \begin{bmatrix}
0 & 0 & 0 \\
0 & \sigma G & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
(26)

The explicit forms of \( [K] \) and \([G]\) are given by

\[
\psi K_{ij} = \int_{\Omega} A_{ij} \psi_{i} \psi_{j} d\Omega,
\]
(27a)

\[
\psi G_{ij} = \int_{\Omega} B_{ij} \psi_{i} \psi_{j} d\Omega,
\]
(27b)

\[
\begin{align*}
& \psi K_{ij} = \int_{\Omega} A_{ij} \psi_{i} \psi_{j} d\Omega, \\
& \psi G_{ij} = \int_{\Omega} B_{ij} \psi_{i} \psi_{j} d\Omega
\end{align*}
\]
(27c)

\[
\begin{align*}
& \psi K_{ij} = \int_{\Omega} A_{ij} \psi_{i} \psi_{j} d\Omega, \\
& \psi G_{ij} = \int_{\Omega} B_{ij} \psi_{i} \psi_{j} d\Omega
\end{align*}
\]
(27d)

\[
\begin{align*}
& \psi K_{ij} = \int_{\Omega} A_{ij} \psi_{i} \psi_{j} d\Omega, \\
& \psi G_{ij} = \int_{\Omega} B_{ij} \psi_{i} \psi_{j} d\Omega
\end{align*}
\]
(27e)

In eqn (24), \( \Delta T \) refers to a buckling temperature and \( \psi \) is the eigenvector of nodal displacements corresponding to an eigenvalue

\[
\Delta = \{u_{i1}, u_{i2}, u'_{i}\}^{T},
\]
(28)

III. NUMERICAL RESULTS

Simply-supported square antisymmetric angle-ply laminates are considered for numerical illustration. The material properties considered in this in-
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Table 1. Effect of side to thickness ratio on nondimensional buckling temperature: $\Delta T = \Delta T_{20} \times 10^4$ for a $\{\pm 45\}$ laminate

<table>
<thead>
<tr>
<th>$l/h$</th>
<th>CLT</th>
<th>FSDT</th>
<th>HSDT(A)</th>
<th>HSDT(B)</th>
<th>LWT</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>19.2875</td>
<td>12.7541</td>
<td>12.6998</td>
<td>12.6991</td>
<td>11.7001</td>
</tr>
<tr>
<td>15</td>
<td>5.4862</td>
<td>4.7885</td>
<td>4.7783</td>
<td>4.7829</td>
<td>4.5226</td>
</tr>
<tr>
<td>20</td>
<td>3.0860</td>
<td>2.8433</td>
<td>2.8485</td>
<td>2.8517</td>
<td>2.7347</td>
</tr>
<tr>
<td>30</td>
<td>1.3716</td>
<td>1.3234</td>
<td>1.3255</td>
<td>1.3224</td>
<td>1.2908</td>
</tr>
<tr>
<td>40</td>
<td>0.7715</td>
<td>0.7560</td>
<td>0.7557</td>
<td>0.7567</td>
<td>0.7439</td>
</tr>
<tr>
<td>50</td>
<td>0.4938</td>
<td>0.4874</td>
<td>0.4873</td>
<td>0.4879</td>
<td>0.4820</td>
</tr>
<tr>
<td>80</td>
<td>0.1929</td>
<td>0.1919</td>
<td>0.1919</td>
<td>0.1921</td>
<td>0.1910</td>
</tr>
<tr>
<td>100</td>
<td>0.1234</td>
<td>0.1230</td>
<td>0.1230</td>
<td>0.1232</td>
<td>0.1227</td>
</tr>
</tbody>
</table>

The results by the present theory, denoted by LWT, provide the most conservative buckling temperature, which is considered to the closest to the three-dimensional elasticity solution.

Table 2 lists the buckling temperatures as a function of a lamination angle and number of layers evaluated using the present theory and other theories. The FSDT and HSDT solutions are taken from Chang [13] with stress-free boundary conditions, and $n$ denotes a wave number in the $x_2$ direction. For $\theta = 10^\circ$ and $15^\circ$, as can be seen in Table 2, FSDT and HSDT results calculated by Chang [13] overestimate the buckling temperature regardless of the number of layers. In view of the results obtained by CLT and present solutions, it is obvious that the buckling mode in these cases is two-half-sine waves, and thus Chang's solutions, which correspond to the buckling temperature of a one-half-sine wave, are found to be invalid. For $\theta = 30^\circ$ and $45^\circ$, the buckling mode is a one-half wave, and the solutions are comparable. The difference between FSDT and HSDT results is negligibly small for all the cases considered.

The effects of the thermal expansion coefficient ratio and the degree of orthotropy are presented for a $\{\pm 45\}$ laminate in Tables 3 and 4, respectively. In

<table>
<thead>
<tr>
<th>Number of layers</th>
<th>Theory</th>
<th>$\theta = 0^\circ$</th>
<th>$15^\circ$</th>
<th>$30^\circ$</th>
<th>$45^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>CLT($n = 1$)</td>
<td>1880.94</td>
<td>1548.34</td>
<td>1436.13</td>
<td>1503.48</td>
</tr>
<tr>
<td></td>
<td>($n = 2$)</td>
<td>1150.14</td>
<td>1257.53</td>
<td>1300.77</td>
<td>1445.76</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>1747.36</td>
<td>1469.92</td>
<td>1380.77</td>
<td>1445.76</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>1747.48</td>
<td>1472.66</td>
<td>1385.37</td>
<td>1451.17</td>
</tr>
<tr>
<td></td>
<td>LWT($n = 1$)</td>
<td>1769.70</td>
<td>1418.90</td>
<td>1310.10</td>
<td>1376.56</td>
</tr>
<tr>
<td></td>
<td>($n = 2$)</td>
<td>1096.52</td>
<td>1193.19</td>
<td>1205.10</td>
<td>1271.56</td>
</tr>
<tr>
<td>4</td>
<td>CLT($n = 1$)</td>
<td>1880.94</td>
<td>2060.78</td>
<td>2558.91</td>
<td>2838.75</td>
</tr>
<tr>
<td></td>
<td>($n = 2$)</td>
<td>1150.14</td>
<td>1630.78</td>
<td>1299.01</td>
<td>1603.75</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>1747.36</td>
<td>1925.50</td>
<td>2387.89</td>
<td>2639.73</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>1747.48</td>
<td>1923.24</td>
<td>2382.64</td>
<td>2633.21</td>
</tr>
<tr>
<td></td>
<td>LWT($n = 1$)</td>
<td>1769.70</td>
<td>1855.30</td>
<td>2230.89</td>
<td>2448.90</td>
</tr>
<tr>
<td></td>
<td>($n = 2$)</td>
<td>1096.52</td>
<td>1481.92</td>
<td>1574.40</td>
<td>1788.90</td>
</tr>
<tr>
<td>8</td>
<td>CLT($n = 1$)</td>
<td>1880.94</td>
<td>2188.90</td>
<td>2839.53</td>
<td>3172.50</td>
</tr>
<tr>
<td></td>
<td>($n = 2$)</td>
<td>1150.14</td>
<td>1724.06</td>
<td>2630.46</td>
<td>2926.04</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>1747.36</td>
<td>2037.14</td>
<td>2630.46</td>
<td>2926.04</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>1747.48</td>
<td>2036.45</td>
<td>2628.69</td>
<td>2923.81</td>
</tr>
<tr>
<td></td>
<td>LWT($n = 1$)</td>
<td>1769.70</td>
<td>2013.54</td>
<td>2573.37</td>
<td>2856.10</td>
</tr>
<tr>
<td></td>
<td>($n = 2$)</td>
<td>1096.52</td>
<td>1374.40</td>
<td>1788.90</td>
<td>2102.50</td>
</tr>
</tbody>
</table>
Table 3. Effect of ratio of thermal expansion coefficients on nondimensional buckling temperature: $\Delta T = \Delta T_{a0} \times 10^3$ for a $[\pm 45]$ laminate, $E_i/E_0 = 30$, $l/h = 10$

<table>
<thead>
<tr>
<th>$\alpha_{22}/\alpha_{11}$</th>
<th>CLT</th>
<th>FSDT</th>
<th>HSDT</th>
<th>LWT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15.0460</td>
<td>10.4431</td>
<td>10.3868</td>
<td>9.3134</td>
</tr>
<tr>
<td>10</td>
<td>11.1730</td>
<td>7.7552</td>
<td>7.7134</td>
<td>6.9163</td>
</tr>
<tr>
<td>20</td>
<td>8.6885</td>
<td>6.0306</td>
<td>5.9980</td>
<td>5.3782</td>
</tr>
<tr>
<td>30</td>
<td>7.1077</td>
<td>4.9334</td>
<td>4.9068</td>
<td>4.3998</td>
</tr>
<tr>
<td>40</td>
<td>6.0138</td>
<td>4.1741</td>
<td>4.1515</td>
<td>3.7225</td>
</tr>
<tr>
<td>50</td>
<td>5.2116</td>
<td>3.6173</td>
<td>3.5978</td>
<td>3.2260</td>
</tr>
</tbody>
</table>

These tables, the HSDT solution is obtained by using Reddy’s theory [12]. It is found that the solution error is not affected by the ratio of thermal expansion coefficients. That is, the HSDT solution differs from the present solution by 11.53% for all the values of $\alpha_{22}/\alpha_{11}$. On the other hand, the difference increases with increasing degrees of orthotropy. For example, the solution error between HSDT and the present solution becomes 14.4% for $E_i/E_0 = 50$, while it is 0.20% for $E_i/E_0 = 2$.

IV. CONCLUDING REMARKS

A layer-wise finite element model is developed to study the thermal buckling of composite plates. The model is capable of predicting accurate buckling temperature of laminated composites. It is found that, while FSDT is fairly in agreement with HSDT for all the cases considered, the difference between the results of the present theory and equivalent single-layer theories is not negligible, and sometimes significant. This is because the coupling effects that come from material anisotropy may be overlooked in equivalent single-layer approaches which are based on averaged properties of a laminate. The layer-wise approach, which models laminates layer-by-layer, is, therefore, preferred for more accurate thermal buckling analysis of laminated composites.

REFERENCES