Flexural–torsional behavior of thin-walled composite box beams using shear-deformable beam theory

Thuc Phuong Vo, Jaehong Lee*

Department of Architectural Engineering, Sejong University, 98 Kunja Dong, Kwangjin Ku, Seoul 143-747, Republic of Korea

Received 15 January 2007; received in revised form 6 December 2007; accepted 6 December 2007
Available online 11 February 2008

Abstract

This paper presents a flexural–torsional analysis of thin-walled composite box beams. A general analytical model applicable to thin-walled composite box beams subjected to vertical and torsional loads is developed. This model is based on the shear-deformable beam theory, and accounts for the flexural–torsional response of the thin-walled composites for an arbitrary laminate stacking sequence configuration, i.e. unsymmetric as well as symmetric. The governing equations are derived from the principle of the stationary value of total potential energy. Numerical results are obtained for thin-walled composites under vertical loading, addressing the effects of fiber angle and span-to-height ratio of the composite beam.

Keywords: Thin-walled composites; Shear deformation; Flexural–torsional response; Finite element method

1. Introduction

Fiber-reinforced composite materials have been used over the past few decades in a variety of structures. Composites have many desirable characteristics, such as high ratio of stiffness and strength to weight, corrosion resistance and magnetic transparency. Thin-walled structural shapes made up of composite materials, which are usually produced by pultrusion, are being increasingly used in many engineering fields. In particular, the use of pultruded composites in civil engineering structures awaits increased attention.

Thin-walled composite structures are often very thin and have complicated material anisotropy. Accordingly, warping and other secondary coupling effects should be considered in the analysis of thin-walled composite structures. The theory of thin-walled closed-section members made of isotropic materials was first developed by Vlasov [1] and Gjelsvik [2]. For fiber-reinforced composites, some analyses have been formulated to analyze composite box beams with varying levels of assumptions. Chandra et al. [3] discussed the structural coupling effects for symmetric and anti-symmetric box beams under flexural, torsional, and extensional loads. Smith and Chopra [4] formulated, and made an evaluation of, an analytical model for composite box beams. The box-beam walls were modeled as orthotropic-ply laminated plates, so that the elastic properties vary both through the thickness and around the box-beam contour; deformation is described in terms of extension, bending, twisting, shearing, and torsion-related out-of-plane warping. Song and Librescu [5] focused on the formulation of the dynamic problem of laminated composite thick- and thin-walled, single-cell beams of arbitrary cross-sections and on the investigation of their associated free vibration behavior. Qin and Librescu [6] provided further contribution and validations on a shear-deformable theory of anisotropic thin-walled beams. The solution methodology was based on the Extended Galerkin’s Method, and the non-classical effects on the static responses and natural frequencies were investigated. Kim and White [7,8] developed an efficient method to account for three-dimensional elastic effects in laminated beam walls. In this analysis primary and secondary torsional warping and transverse shear effects, both of the cross-section and of the beam walls, were considered. Pluzsik and Kollar [9] presented a beam theory for thin-walled open- and closed-section composite beams which analyzed the effect of shear deformation and restrained warping. Salim and Davalos [10] presented the
linear analysis of open and closed sections made of general laminated composites by extending Gjelsvik’s model [2]. This model accounted for all possible elastic couplings in composite sections, such as extension and bending torsion. The effect of warping torsion on the torsional stiffness of the beam was investigated. Recently, Librescu and Song [11] introduced the monograph about thin-walled composite beams. The monograph was concerned not only with the foundation and formulation of modern linear and non-linear theories of composite thin-walled beams but also provided powerful mathematical tools to address issues of statics and dynamics of composite thin-walled beams. The effects of transverse shear, warping inhibition, and of various elastic couplings on the behavior of these structures, have been highlighted. Piovan and Cortinez [12] presented a new theoretical model for the generalized linear analysis of thin-walled beams with open or closed cross-sections. The model was developed by employing a non-linear displacement field and allowed for the study of many problems of static, free vibrations with or without arbitrary initial stresses and linear stability of composite thin-walled beams with general cross-sections. More recently, Vo and Lee [13] presented an analytical model which accounted for flexural–torsional behavior of composite box beams. They developed one-dimensional finite element model to investigate the flexural–torsional behavior of thin-walled composite box beams.

In this paper, a general analytical model for thin-walled open-section composite beams developed by Lee [14] has been extended to the composite box beams. This model is based on the first-order shear deformable beam theory, and accounts for all the structural coupling coming from the material anisotropy. Governing equations are derived from the principle of the stationary value of the total potential energy. Numerical results are obtained for thin-walled composites under vertical loading, addressing the effects of fiber angle and span-to-height ratio of the composite beams.

2. Kinematics

The theoretical developments presented in this paper require two sets of coordinate systems which are mutually interrelated. The first coordinate system is the orthogonal Cartesian coordinate system \((x, y, z)\), for which the \(x\) and \(y\) axes lie in the plane of the cross-section and the \(z\) axis, parallel to the longitudinal axis of the beam. The second coordinate system is the local plate coordinate \((n, s, z)\) as shown in Fig. 1, wherein the \(n\) axis is normal to the middle surface of a plate element, the \(s\) axis is tangent to the middle surface and is directed along the contour line of the cross-section. The \((n, s, z)\) and \((x, y, z)\) coordinate systems are related through an angle of orientation \(\theta\) as defined in Fig. 1. Point \(P\) is called the pole axis, through which the axis parallel to the \(z\) axis is called the pole axis.

To derive the analytical model for a thin-walled composite beam, the following assumptions are made

1. The contour of the thin wall does not deform in its own plane.

2. Transverse shear strains \(\gamma_{xz}^o, \gamma_{yz}^o\) and warping shear \(\gamma_{\omega}^o\) are incorporated. It is assumed that they are uniform over the cross-sections.

3. The linear shear strain \(\gamma_{xz}\) of the middle surface will have the same distribution in the contour direction as it has in the St. Venant torsion in each element.

According to assumption 1, the mid-surface displacement \(\bar{u}, \bar{v}\) at a point \(A\) in the contour coordinate system can be expressed in terms of displacements \(U, V\) of the pole \(P\) in the \(x, y\) directions, respectively, and the rotation angle \(\Phi\) about the pole axis

\[
\bar{u}(s, z) = U(z) \sin \theta(s) - V(z) \cos \theta(s) - \Phi(z) q(s) \tag{1a}
\]

\[
\bar{v}(s, z) = U(z) \cos \theta(s) + V(z) \sin \theta(s) + \Phi(z) r(s). \tag{1b}
\]

These equations apply to the whole contour. The out-of-plane shell displacement \(\bar{w}\) can now be found from assumption 2. For each element of the middle surface, the mid-surface shear strains in the contour can be expressed with respect to the transverse shear and the warping shear strains

\[
\vec{\gamma}_{xz}(s, z) = \gamma_{xz}^o(z) \cos \theta(s) + \gamma_{yz}^o(z) \sin \theta(s)
\]

\[
- \gamma_{\omega}^o(z) r(s) - \left[\gamma_{\omega}^o(z) - \Phi'(z)\right] F(s) t(s) \tag{2b}
\]

where \(t(s)\) is the thickness of contour box section, \(F(s)\) is the St. Venant circuit shear flow. Further, it is assumed that mid-surface shear strain in \(s-n\) direction is zero (\(\vec{\gamma}_{sn} = 0\)). From the definition of the shear strain, \(\vec{\gamma}_{xz}\) can also be given for each element of middle surface as

\[
\vec{\gamma}_{xz}(s, z) = \frac{\partial \vec{v}}{\partial z} + \frac{\partial \bar{w}}{\partial s}. \tag{3}
\]

After substituting for \(\vec{v}\) from Eq. (1) into Eq. (3) and considering the following geometric relations

\[
dx = ds \cos \theta \tag{4a}
\]

\[
dy = ds \sin \theta. \tag{4b}
\]

Displacement \(\bar{w}\) can be integrated with respect to \(s\) from the origin to an arbitrary point on the contour

\[
\bar{w}(s, z) = W(z) + \Psi_y(z)x(s) + \Psi_x(z)y(s) + \Psi_{\omega}(z) \omega(s) \tag{5}
\]
where $\Psi_x$, $\Psi_y$, and $\Psi_\omega$ represent rotations of the cross-section with respect to $x$, $y$ and $\omega$, respectively, given by

$$
\Psi_y = \gamma_y^0(z) - U' \quad (6a)
$$

$$
\Psi_x = \gamma_x^0(z) - V' \quad (6b)
$$

$$
\Psi_\omega = \gamma_\omega^0(z) - \Phi'. \quad (6c)
$$

When the transverse shear effect is ignored, Eq. (6) degenerates into

$$
\Psi_x = -U', \; \Psi_y = -V' \; \text{and} \; \Psi_\omega = -\Phi', \quad (7a)
$$

as a result, the number of unknown variables reduces to four leading to the Euler–Bernoulli beam model. The prime (') is used to indicate differentiation with respect to $z$; and $\omega$ is the so-called sectorial coordinate or warping function given by

$$
\omega(s) = \int_{s_0}^s \left[ r(s) - \frac{F(s)}{t(s)} \right] ds \quad (7a)
$$

$$
\int_{s_0}^s \frac{F(s)}{t(s)} ds = 2 A_i \quad i = 1, \ldots, n \quad (7b)
$$

where $r(s)$ is height of a triangle with the base $ds$; $A_i$ is the area circumscribed by the contour of the $i$ circuit. The explicit forms of $\omega(s), F(s)$ for the box section are given in the Appendix of Ref. [13].

The displacement components $u, v, w$ representing the deformation of any generic point on the profile section are given with respect to the mid-surface displacements $\bar{u}, \bar{v}, \bar{w}$ by assuming the first-order variation of in-plane displacements $v, w$ through the thickness of the contour as

$$
u(s, z, n) = \bar{u}(s, z) \quad (8a)$$

$$
v(s, z, n) = \bar{v}(s, z) + n \bar{\psi}_z(s, z) \quad (8b)$$

$$w(s, z, n) = \bar{w}(s, z) + n \bar{\psi}_z(s, z) \quad (8c)$$

where, $\bar{\psi}_z$ and $\bar{\psi}_z$ denote the rotations of a transverse normal about the $z$ and $s$ axes, respectively. These functions can be determined by considering that the mid-surface shear strain $\gamma_{nz}$ is given by the definition

$$
\bar{\gamma}_{nz}(s, z) = \frac{\partial \bar{w}}{\partial n} + \frac{\partial \bar{u}}{\partial z}. \quad (9)
$$

By comparing Eqs. (2) and (9), the function $\bar{\psi}_z$ can be written as

$$
\bar{\psi}_z = \Psi_z \sin \theta - \Psi_s \cos \theta - \Psi_\omega q. \quad (10)
$$

Similarly, using the assumption that the shear strain $\gamma_{sn}$ should vanish at mid-surface, the function $\bar{\psi}_s$ can be obtained as

$$
\bar{\psi}_s = -\frac{\partial \bar{u}}{\partial s}. \quad (11)
$$

The strains associated with the small-displacement theory of elasticity are given by

$$
\epsilon_z(s, z, n) = \bar{\epsilon}_z(s, z) + n \bar{k}_z(s, z) \quad (12a)
$$

$$
\epsilon_z(s, z, n) = \bar{\epsilon}_z(s, z) + n \bar{k}_z(s, z) \quad (12b)
$$

$$
\gamma_{sz}(s, z, n) = \bar{\gamma}_{sz}(s, z) + n \bar{k}_{sz}(s, z) \quad (12c)
$$

$$
\gamma_{nz}(s, z, n) = \bar{\gamma}_{nz}(s, z) + n \bar{k}_{nz}(s, z) \quad (12d)
$$

where

$$
\bar{\epsilon}_z = \frac{\partial \bar{v}}{\partial s}, \quad \bar{\epsilon}_s = \frac{\partial \bar{w}}{\partial z} \quad (13a)
$$

$$
\bar{k}_z = \frac{\partial \bar{\psi}_z}{\partial s}; \quad \bar{k}_z = \frac{\partial \bar{\psi}_z}{\partial z} \quad (13b)
$$

$$
\bar{k}_{sz} = \frac{\partial \bar{\psi}_z}{\partial s} + \frac{\partial \bar{\psi}_s}{\partial z}; \quad \bar{k}_{nz} = 0. \quad (13c)
$$

All the other strains are identically zero. In Eq. (13), $\bar{\epsilon}_z$ and $\bar{k}_z$ are assumed to be zero, $\bar{\epsilon}_s$, $\bar{\psi}_z$ and $\bar{\psi}_z$ are the mid-surface axial strain and biaxial curvature of the shell, respectively. The above shell strains can be converted to beam strain components by substituting Eqs. (1), (5) and (8) into Eq. (13) as

$$
\epsilon_z = \epsilon_z^0 + x\kappa_y + y\kappa_x + \omega\kappa_\omega \quad (14a)
$$

$$
\kappa_z = \kappa_z^0 \quad (15a)
$$

$$
\kappa_y = \kappa_y^0 \quad (15b)
$$

$$
\kappa_x = \kappa_x^0 \quad (15c)
$$

$$
\kappa_\omega = \kappa_\omega^0 \quad (15d)
$$

$$
\kappa_{sz} = \kappa_{sz}^0 \quad (15e)
$$

The resulting strains can be obtained from Eqs. (12) and (14) as

$$
\epsilon_z = \epsilon_z^0 + (x + n \sin \theta)\kappa_y \quad (16a)
$$

$$
\gamma_{sz} = \gamma_{sz}^0 \cos \theta \quad (16b)
$$

$$
\gamma_{nz} = \gamma_{nz}^0 \sin \theta - \gamma_{nz}^0 \cos \theta - \gamma_{nz}^0 q. \quad (16c)
$$

3. Variational formulation

The total potential energy of the system is calculated by the sum of the strain energy and the work done by an external force

$$
\Pi = U + V \quad (17)
$$

where $U$ is the strain energy

$$
U = \frac{1}{2} \int_{s_0}^s (\sigma_z \epsilon_z + \sigma_{s_z} \gamma_{sz} + \sigma_{n_z} \gamma_{nz}) ds. \quad (18)
$$

The strain energy is calculated by substituting Eq. (16) into Eq. (18)
\[ U = \frac{1}{2} \int \left\{ \sigma_z [e_z^2 + (x + n \sin \theta) \kappa_y + (y - n \cos \theta) \kappa_x \\
+ (\omega - n q) \kappa_{\omega} ] + \sigma_{yz} \left[ \frac{\gamma_{yz}^0 \cos \theta}{2t} \\
+ \gamma_{yz}' \sin \theta + \gamma_{yz}^0 \left( r - \frac{F}{2t} \right) + \kappa_{\omega} \left( n + \frac{F}{2t} \right) \right] \\
+ \sigma_{nz} \left[ \frac{\gamma_{nz}^0 \cos \theta - \gamma_{nz}^0 \cos \theta - \gamma_{nz}^0 \cos \theta}{} \right] \right\} dV. \] (19)

The variation of the strain energy, Eq. (19), can be stated as
\[ \delta U = \int_0^l \left( N_z \delta \varepsilon_z + M_y \delta \kappa_y + M_x \delta \kappa_x + M_{\omega} \delta \kappa_{\omega} \\
+ V_x \delta \gamma_{xz}^0 + V_y \delta \gamma_{yz}^0 + T \delta \gamma_{\omega}^0 + M_{\omega} \delta \gamma_{\omega} \right) ds \] (20)

where \( N_z, M_x, M_y, M_{\omega}, V_x, V_y, T, M_{\omega} \) are the axial force, bending moments in the \( x \) and \( y \) directions, warping moment (bimoment), shear force in the \( x \) and \( y \) directions, and torsional moments, respectively, defined by integrating over the cross-sectional area \( A \) as
\[ N_z = \int_A \sigma_z ds \] (21a)
\[ M_x = \int_A \sigma_z (x + n \sin \theta) ds \] (21b)
\[ M_y = \int_A \sigma_z (y - n \cos \theta) ds \] (21c)
\[ M_{\omega} = \int_A \sigma_z (\omega - n q) ds \] (21d)
\[ V_x = \int_A \left( \sigma_{xz} \cos \theta + \sigma_{nz} \sin \theta \right) ds \] (21e)
\[ V_y = \int_A \left( \sigma_{yz} \sin \theta - \sigma_{nz} \cos \theta \right) ds \] (21f)
\[ T = \int_A \left[ \sigma_z \left( r - \frac{F}{2t} \right) - \sigma_{nz} q \right] ds \] (21g)
\[ M_{\omega} = \int_A \sigma_z \left( n + \frac{F}{2t} \right) ds. \] (21h)

The variation of the work done by the vertical and torsional loads can be stated as
\[ \delta V = - \int_0^l \left( \nabla \delta V + T \delta \Phi \right) dz \] (22)

where \( \nabla \) is the vertical load and \( T \) is the applied torsional load. Using the principle that the variation of the total potential energy is zero, the following weak statement is obtained
\[ 0 = \int_0^l \left( N_z \delta W' + M_y \delta \Psi_y + M_x \delta \Psi_x \\
+ M_{\omega} \delta \Psi_{\omega} \right. \\
+ V_x \delta \left( U' + \Psi_x \right) + V_y \delta \left( V' + \Psi_y \right) + T \delta \left( \Phi' + \Psi_{\omega} \right) + M_{\omega} \delta \left( \Phi' - \Psi_{\omega} \right) \\
+ V_y \delta V + T \delta \Phi \right) ds. \] (23)

4. Constitutive equations

The constitutive equations of a \( k \)th orthotropic lamina in the laminate coordinate system of the box section are given by
\[ \begin{bmatrix} \varepsilon_z \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11}^{k} & \bar{Q}_{16}^{k} \\ \bar{Q}_{16}^{k} & \bar{Q}_{56}^{k} \end{bmatrix} \begin{bmatrix} \varepsilon_z \\ \gamma_{yz} \end{bmatrix} \] (24)

where \( \bar{Q}_{ij}^{k} \) are transformed reduced stiffnesses. The transformed reduced stiffnesses can be calculated from the transformed stiffnesses based on the plane stress assumption and plane strain assumption. A more detailed explanation can be found in Ref. [15].

The constitutive relation for out-of-plane stress and strain is given by
\[ \sigma_{nz} = \bar{Q}_{55} \gamma_{nz}. \] (25)

The constitutive equations for bar forces and bar strains are obtained by using Eqs. (16), (21) and (24)
\[ \begin{bmatrix} N_z \\ M_y \\ M_x \\ M_{\omega} \\ V_x \\ V_y \\ T \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{14} & E_{15} & E_{16} & E_{17} & E_{18} \\ E_{22} & E_{23} & E_{24} & E_{25} & E_{26} & E_{27} & E_{28} \\ E_{33} & E_{34} & E_{35} & E_{36} & E_{37} & E_{38} & E_{39} \\ E_{44} & E_{45} & E_{46} & E_{47} & E_{48} & E_{49} & E_{50} \\ E_{55} & E_{56} & E_{57} & E_{58} & E_{59} & E_{60} & E_{61} \\ E_{66} & E_{67} & E_{68} & E_{69} & E_{70} & E_{71} & E_{72} \\ E_{77} & E_{78} & E_{79} & E_{80} & E_{81} & E_{82} & E_{83} \end{bmatrix} \begin{bmatrix} \varepsilon_z \\ \kappa_y \\ \kappa_x \\ \kappa_{\omega} \\ \gamma_{xz} \end{bmatrix} \] (26)

where \( E_{i,j} \) are stiffnesses of the thin-walled composite. \( (E_{i,j}, E_{i,8} i = 1 \ldots 8) \) can be defined by
\[ E_{15} = \int_s (A_{16} \frac{F}{2t} + B_{16}) ds \] (27a)
\[ E_{18} = \int_s A_{16} \left( r - \frac{F}{2t} \right) ds \] (27b)
\[ E_{25} = \int_s \left[ A_{16} \frac{F}{2t} x + B_{16} \right. \\
\left. \times \left( x + \frac{F \sin \theta}{2t} \right) + D_{16} \sin \theta \right] ds \] (27c)
where \( A_{ij}, B_{ij} \) and \( D_{ij} \) matrices are extensional, coupling and bending stiffnesses, respectively, defined by

\[
(A_{ij}, B_{ij}, D_{ij}) = \int \tilde{Q}_{ij}(1, n, n^2)dn.
\]

Other values of \( E_{ij} \) can be found in Ref. [14]. The explicit forms of the laminate stiffnesses \( E_{ij} \) can be calculated for the composite box section as given in the Appendix.

5. Governing equations

The equilibrium equations of the present study can be obtained by integrating the derivatives of the varied quantities by parts and collecting the coefficients of \( \delta W, \delta U, \delta V, \delta \Phi, \delta \Psi_{y}, \delta \Psi_{x} \) and \( \delta \Psi_{\omega} \)

\[
N'_{z} = 0 \quad (29a)
\]

\[
V'_{x} = 0 \quad (29b)
\]

\[
V'_{y} = V_{y} \quad (29c)
\]

\[
M'_{x} + T' = T \quad (29d)
\]

\[
M'_{y} = M_{y} \quad (29e)
\]

\[
M'_{\omega} + M_{\omega} - T = 0. \quad (29f)
\]

The natural boundary conditions are of the form

\[
\delta W : N_{z} \quad (30a)
\]

\[
\delta U : V_{x} \quad (30b)
\]

\[
\delta V : V_{y} \quad (30c)
\]

\[
\delta \Phi : T + M_{x} \quad (30d)
\]

\[
\delta \Psi_{y} : M_{y} \quad (30e)
\]

\[
\delta \Psi_{x} : M_{x} \quad (30f)
\]

\[
\delta \Psi_{\omega} : M_{\omega}. \quad (30g)
\]

The 7th one denotes the warping restraint boundary condition. When the warping of the cross-section is restrained, \( \Psi_{\omega} = 0 \) and when the warping is not restrained, \( M_{\omega} = 0 \). By substituting Eqs. (15) and (26) into Eq. (29) the explicit form of the governing equations can be expressed with respect to the laminate stiffnesses \( E_{ij} \) as
are involved, and these equations are well
\begin{equation}
\psi_x + E_{34} \psi_y' + (E_{35} - E_{38} + E_{47}) \psi_x' + (E_{57} - E_{78}) \psi_x = \frac{E_s}{E} \psi_s = 0.
\end{equation}

Eq. (31) is the most general form of a thin-walled laminated composite with a box section. For general anisotropic materials, the dependent variables, \(U, V, W, \phi, \psi_x, \psi_y\), and \(\psi_{wo}\) are fully coupled implying that the beam undergoes a coupled behavior involving bending, twisting, extension, transverse shearing, and warping. If all the coupling effects are neglected, Eq. (31) can be simplified to the uncoupled differential equations as

\begin{align}
(EA)_{\text{com}} W'' &= 0 \quad \text{(32a)} \\
(GA)_x (U'' + \psi_y') &= 0 \quad \text{(32b)} \\
(GA)_x (V'' + \psi_y') &= 0 \quad \text{(32c)} \\
[(GJ)_x + (GJ)_y] \psi'' - (GJ)_x \psi_x' &= T \quad \text{(32d)} \\
(EI)_y \psi_y - (GA)_x (U' + \psi_y) &= 0 \quad \text{(32e)} \\
(EI)_x \psi_x - (GA)_x (V' + \psi_x) &= 0 \quad \text{(32f)} \\
[(GJ)_y + (GJ)_x] \psi'' + (GJ)_y \psi_y' &= 0 \quad \text{(32g)}
\end{align}

From the above equations, \((EA)_{\text{com}}\) represents axial rigidity; \((GA)_x\), \((GA)_y\) represent shear rigidities with respect to \(x\) and \(y\) axis; \((EI)_y\) and \((EI)_x\) represent flexural rigidities with respect to \(x\) and \(y\) axes; \((EJ)_x\) represents warping rigidity; and \((GJ)_x\), \((GJ)_y\), \((GJ)_3\) represent torsional rigidities of the thin-walled composite, respectively, written as

\begin{align}
(EA)_{\text{com}} &= E_{11} \quad \text{(33a)} \\
(EI)_y &= E_{22} \quad \text{(33b)} \\
(EI)_y &= E_{22} \quad \text{(33c)} \\
(EI)_x &= E_{22} \quad \text{(33d)} \\
(GA)_x &= E_{44} \quad \text{(33e)} \\
(GA)_x &= E_{44} \quad \text{(33f)} \\
(GJ)_x &= E_{55} + E_{68} \quad \text{(33g)} \\
(GJ)_y &= E_{55} - E_{68} \quad \text{(33h)} \\
(GJ)_3 &= 2E_{58} \quad \text{(33i)}
\end{align}

For the bending analysis with respect to \(x\)-axis, only Eqs. (32c) and (32f) are involved, and these equations are well known as Timoshenko beam equations.

6. Finite element formulation

The present theory for thin-walled composite beams described in the previous section was implemented via a one-dimensional displacement-based finite element method. The generalized displacements are expressed over each element as a linear combination of the one-dimensional Lagrange interpolation function \(\phi_j\) associated with node \(j\) and the nodal values

\begin{equation}
W = \sum_{j=1}^{n} w_j \phi_j \quad \text{(34a)}
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig_2.png}
\caption{Geometry and stacking sequences of thin-walled composite box beam for verification.}
\end{figure}

\begin{equation}
U = \sum_{j=1}^{n} u_j \phi_j \quad \text{(34b)}
\end{equation}

\begin{equation}
V = \sum_{j=1}^{n} v_j \phi_j \quad \text{(34c)}
\end{equation}

\begin{equation}
\phi = \sum_{j=1}^{n} \phi_j \phi_j \quad \text{(34d)}
\end{equation}

\begin{equation}
\psi_y = \sum_{j=1}^{n} \psi_{x,j} \phi_j \quad \text{(34e)}
\end{equation}

\begin{equation}
\psi_x = \sum_{j=1}^{n} \psi_{x,j} \phi_j \quad \text{(34f)}
\end{equation}

\begin{equation}
\psi_{wo} = \sum_{j=1}^{n} \psi_{wo,j} \phi_j \quad \text{(34g)}
\end{equation}

Substituting these expressions into the weak statement in Eq. (23), the finite element model of a typical element can be expressed as

\begin{equation}
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & w \\
K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & \vdots & u \\
K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & \vdots & \vdots & v \\
K_{44} & K_{45} & K_{46} & K_{47} & \vdots & \vdots & \vdots & \vdots \\
K_{55} & K_{56} & K_{57} & \vdots & \vdots & \vdots & \vdots & \psi_y \\
K_{66} & K_{67} & \vdots & \vdots & \vdots & \vdots & \vdots & \psi_x \\
K_{77} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \psi_{wo}
\end{bmatrix}
= \begin{bmatrix}
f_3 \\
f_4 \\
f_5 \\
f_6 \\
f_7 \\
f_8 \\
0 \\
0
\end{bmatrix}
\end{equation}

where \([K]\) is the element stiffness matrix and \([f]\) is the element force vector. A more detailed explanation for the explicit forms of \([K]\) and \([f]\) can be found in Ref. [14].

7. Numerical examples

For the purpose of verification, a cantilever composite box beam with length \(l = 0.762\ m\), the cross-section and the stacking sequences shown in Fig. 2 is subjected to a 4.45 N
The resulting values of bending slope and the angle of twist using the present analysis are compared with the previous available results for two stacking sequences CAS, CUS in Figs. 3–5. It is seen that the results from the present analysis are in good agreement with the solution in Ref. [4,6,8].

In order to investigate the coupling, the transverse shear deformation and warping restraint effects, a clamped composite box beam under an eccentric uniform load $V_y = -6.5 \text{ kN/m}$ is considered (Fig. 6). The loads with respect to shear center are $V_y = -6.5 \text{ kN/m}$ and $T = -0.325 \text{ kN m/m}$. For convenience, the following non-dimensional values of angle of twist, vertical displacement and shear-deformation parameters are used

$$\bar{\phi} = \frac{\phi E_2 b_3^3}{V_y l^3}$$

$$\bar{\psi} = \frac{\psi E_2 b_3^3}{V_y l^3}$$

$$\alpha = \frac{v}{\bar{\psi}}$$

where $v_s$ is the vertical displacement due to the shear deformation.

In Fig. 7, the shear-deformation parameter $\alpha$ with respect to the span-to-height ratio for different symmetric and unsymmetric lay-ups are compared with Ref. [9]. It is seen that all the results are in excellent agreement.

Two layers with equal thicknesses are considered as an anti-symmetric angle-ply laminate $[\theta/-\theta]$ in the flanges and webs (Fig. 8(a)). By using the warping restraint (WR) and free warping (FW) models, the maximum angle of twist and the vertical displacement at mid-span of the beam with respect to the fiber angle change are shown in Figs. 9 and 10 for $l/b_1 = 10$ and $l/b_1 = 50$. In generating Figs. 9 and 10, the finite element solution with no shear effects is calculated based on a previous research [13]. The angle of twist is not affected by shear deformation as shown in Fig. 9 even for a lower span-
Table 1  
Ratio of coupling stiffnesses with respect to the bending stiffness and shear-deformation parameter $\alpha$ when flanges and webs are all anti-symmetric angle-ply

<table>
<thead>
<tr>
<th>Fiber angle</th>
<th>$E_{15}/E_{33}$</th>
<th>$E_{27}/E_{33}$</th>
<th>$E_{36}/E_{33}$</th>
<th>$E_{48}/E_{33}$</th>
<th>Shear-deformation parameter $\alpha$</th>
<th>Ratio $l/b_1 = 10$</th>
<th>Ratio $l/b_1 = 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.684</td>
<td>0.080</td>
</tr>
<tr>
<td>15</td>
<td>−0.044</td>
<td>−0.029</td>
<td>0.015</td>
<td>0.000</td>
<td>0.000</td>
<td>0.528</td>
<td>0.043</td>
</tr>
<tr>
<td>30</td>
<td>−0.083</td>
<td>−0.055</td>
<td>0.028</td>
<td>0.000</td>
<td>0.000</td>
<td>0.266</td>
<td>0.014</td>
</tr>
<tr>
<td>45</td>
<td>−0.076</td>
<td>−0.051</td>
<td>0.025</td>
<td>0.000</td>
<td>0.000</td>
<td>0.136</td>
<td>0.006</td>
</tr>
<tr>
<td>60</td>
<td>−0.024</td>
<td>−0.016</td>
<td>0.008</td>
<td>0.000</td>
<td>0.000</td>
<td>0.112</td>
<td>0.005</td>
</tr>
<tr>
<td>75</td>
<td>−0.001</td>
<td>−0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.119</td>
<td>0.005</td>
</tr>
<tr>
<td>90</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.124</td>
<td>0.006</td>
</tr>
</tbody>
</table>

Table 2  
Ratio of coupling stiffnesses with respect to the bending stiffness and shear-deformation parameter $\alpha$ when the bottom flange and the right web are unidirectional while the top flange and the left web are symmetric angle-ply

<table>
<thead>
<tr>
<th>Fiber angle</th>
<th>$E_{23}/E_{33}$</th>
<th>$E_{56}/E_{33}$</th>
<th>$E_{57}/E_{33}$</th>
<th>$E_{68}/E_{33}$</th>
<th>$E_{78}/E_{33}$</th>
<th>Shear-deformation parameter $\alpha$</th>
<th>Ratio $l/b_1 = 10$</th>
<th>Ratio $l/b_1 = 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.684</td>
<td>0.080</td>
<td>0.080</td>
</tr>
<tr>
<td>15</td>
<td>0.002</td>
<td>−0.057</td>
<td>−0.115</td>
<td>−0.129</td>
<td>−0.085</td>
<td>0.644</td>
<td>0.070</td>
<td>0.070</td>
</tr>
<tr>
<td>30</td>
<td>0.055</td>
<td>−0.136</td>
<td>−0.273</td>
<td>−0.393</td>
<td>−0.377</td>
<td>0.597</td>
<td>0.058</td>
<td>0.058</td>
</tr>
<tr>
<td>45</td>
<td>0.147</td>
<td>−0.099</td>
<td>−0.199</td>
<td>−0.380</td>
<td>−0.462</td>
<td>0.506</td>
<td>0.045</td>
<td>0.045</td>
</tr>
<tr>
<td>60</td>
<td>0.178</td>
<td>−0.041</td>
<td>−0.082</td>
<td>−0.235</td>
<td>−0.347</td>
<td>0.445</td>
<td>0.037</td>
<td>0.037</td>
</tr>
<tr>
<td>75</td>
<td>0.182</td>
<td>−0.009</td>
<td>−0.019</td>
<td>−0.147</td>
<td>−0.265</td>
<td>0.415</td>
<td>0.032</td>
<td>0.032</td>
</tr>
<tr>
<td>90</td>
<td>0.182</td>
<td>0.000</td>
<td>0.000</td>
<td>−0.120</td>
<td>−0.239</td>
<td>0.406</td>
<td>0.027</td>
<td>0.027</td>
</tr>
</tbody>
</table>

Fig. 7. Shear-deformation parameter ($\alpha = v_3/v$) with respect to span-to-height change on a clamped composite box beam under an eccentric uniform load with orthotropic lay-up.

In Eq. (38), the first term denotes the displacement by the classical beam theory, and the second term is the displacement by the shear deformation. For this stacking sequence, the coupling stiffnesses $E_{15}$, $E_{27}$, $E_{36}$ and $E_{48}$ do not vanish while all the other coupling stiffnesses become zero. That is, the orthotropy solution given in Eq. (38) might not be accurate. However, since the coupling stiffnesses are very small compared to the bending stiffness $E_{33}$ (Table 1), the coupling effects coming from the material anisotropy become negligible. Consequently, the finite element solution with warping restraint (WR), free warping (FW) models and the simple orthotropy solution of the classical beam theory agree well as shown in Fig. 10.

To investigate the coupling and transverse shear effects further, the same configuration with the previous example except
the laminate stacking sequence is considered. The stacking sequence of the top flange and the left web is $[\theta/ - \theta]_s$, while the bottom flange and the right web are assumed unidirectional (Fig. 8b). For this stacking sequence, the coupling stiffnesses $E_{23}$, $E_{24}$, $E_{25}$, $E_{34}$, $E_{35}$, $E_{45}$, $E_{56}$, $E_{57}$, $E_{58}$, $E_{68}$ and $E_{78}$ do not vanish while all the other coupling stiffnesses become zero. Especially, $E_{23}$, $E_{56}$, $E_{57}$, $E_{68}$ and $E_{78}$ become no more negligibly small as given in Table 2. The shear-deformation parameter $\alpha$ in Table 2 shows that the shear effects are significant even for a higher fiber angle for $l/b_1 = 10$. For a lower span-to-height ratio (Fig. 11), however, the solutions excluding shear effects remarkably underestimate the displacement for all the ranges of the fiber angle. The orthotropy solutions disagree with the finite element solutions as the anisotropy of the beam gets higher and fiber angle increases. For $l/b_1 = 50$, as the fiber angle increases, the orthotropy solution and the finite element solution show discrepancy indicating that the coupling effects become significant (Fig. 12). The shear effects are negligible in this case.

8. Concluding remarks

A general analytical model is developed to study the flexural–torsional behavior of thin-walled composite box beams. The model is capable of predicting accurate deflection for various configurations including boundary conditions, laminate orientation and span-to-height ratio. To formulate the problem, a one-dimensional displacement-based finite element method is employed. The shear effects become significant for lower span-to-height ratio and higher degrees of orthotropy of the beam. The orthotropy solution is accurate for lower degrees of material anisotropy, but, becomes inappropriate as the anisotropy of the beam gets higher, and fully-coupled equations should be considered for accurate analysis of thin-walled
composite beams. The present analytical model is found to be appropriate and efficient in analyzing the flexural–torsional problem of thin-walled composite box beams.

Acknowledgment

The support provided for this research by the Korea Ministry of Construction and Transportation through Grant 2002-F02-01 is gratefully acknowledged.

Appendix

The explicit forms of the laminate stiffnesses $E_{ij}$ for the composite box section in Fig. 13 can be defined by

$$E_{16} = A_{16}^2 b_2 - A_{16}^4 b_2$$

$$E_{17} = -A_{16}^2 b_1 + A_{16}^4 b_1$$

$$E_{18} = A_{16}^4 \left( -x_1 + x_p - \frac{F_{21l}}{2} \right) b_1$$

$$+ A_{16}^2 \left( -y_2 + y_p - \frac{F_{21l}}{2} \right) b_2$$

$$+ A_{16}^4 \left( x_3 - x_p - \frac{F_{21t}}{2} \right) b_1$$

$$+ A_{16}^4 \left( y_4 - y_p - \frac{F_{21t}}{2} \right) b_2$$

$$E_{26} = \frac{1}{2} A_{16}^2 b_2 + A_{16}^2 x_1 b_2 + \frac{1}{2} A_{16}^4 b_2 - A_{16}^4 x_3 b_2$$

$$E_{27} = -A_{16}^4 x_1 b_1 + B_{16}^4 b_1 + A_{16}^4 x_3 b_1 + B_{16}^4 b_1$$

$$E_{28} = (A_{16}^4 x_1 - B_{16}^4) \left( -x_1 + x_p - \frac{F_{21l}}{2} \right) b_1$$

$$+ \frac{1}{2} A_{16}^2 \left( -y_2 + y_p - \frac{F_{21l}}{2} \right) b_2$$

$$+ A_{16}^4 x_1 \left( -y_2 + y_p - \frac{F_{21t}}{2} \right) b_2$$

$$+ (A_{16}^4 x_3 + B_{16}^4) \left( x_3 - x_p - \frac{F_{21t}}{2} \right) b_1$$

$$- \frac{1}{2} A_{16}^4 \left( y_4 - y_p - \frac{F_{21t}}{2} \right) b_2$$

$$+ A_{16}^4 x_3 \left( y_4 - y_p - \frac{F_{21t}}{2} \right) b_2$$

$$E_{36} = A_{16}^2 y_2 b_2 - B_{16}^2 b_2 - A_{16}^2 y_4 b_2 - B_{16}^4 b_2$$

$$E_{37} = \frac{1}{2} A_{16}^4 b_1^2 - A_{16}^4 y_4 b_1 + \frac{1}{2} A_{16}^3 b_1^2 + A_{16}^3 y_2 b_1$$

$$\quad E_{38} = -\frac{1}{2} A_{16}^4 \left( -x_1 + x_p - \frac{F_{21l}}{2} \right) b_1$$

$$\quad + A_{16}^4 y_4 \left( -x_1 + x_p - \frac{F_{21l}}{2} \right) b_1$$

$$\quad + (A_{16}^2 y_2 - B_{16}^2) \left( -y_2 + y_p - \frac{F_{21t}}{2} \right) b_2$$

$$\quad + \frac{1}{2} A_{16}^3 \left( x_3 - x_p - \frac{F_{21t}}{2} \right) b_1$$

$$\quad + A_{16}^4 y_2 \left( x_3 - x_p - \frac{F_{21t}}{2} \right) b_1$$

$$\quad + (A_{16}^4 y_4 + B_{16}^4) \left( y_4 - y_p - \frac{F_{21t}}{2} \right) b_2$$

$$\quad E_{46} = \frac{1}{2} (A_{16}^2 A_2 - B_{16}^2) b_2^2 + A_{16}^2 (A_1 b_1 + C) b_2$$

$$\quad + \frac{1}{2} (-A_{16}^4 A_4 + B_{16}^4 b_2^2$$

$$\quad - A_{16}^4 (C + A_1 b_1 + A_2 b_2 + A_3 b_1) b_2$$

$$\quad E_{47} = \frac{1}{2} (-A_{16}^4 A_1 + B_{16}^4 r_1^2 - A_{16}^4 C b_1$$

$$\quad + \frac{1}{2} (A_{16}^4 A_3 - B_{16}^4 b_1^2$$

$$\quad + A_{16}^4 (A_1 b_1 + A_2 b_2 + C) b_1$$

$$\quad E_{48} = \frac{1}{2} (A_{16}^4 A_1 - B_{16}^4) \left( -x_1 + x_p - \frac{F_{21l}}{2} \right) b_1$$

$$\quad + A_{16}^4 C \left( -x_1 + x_p - \frac{F_{21l}}{2} \right) b_1$$

$$\quad + \frac{1}{2} (A_{16}^4 A_2 - B_{16}^4) \left( -y_2 + y_p - \frac{F_{21t}}{2} \right) b_2$$

$$\quad + A_{16}^4 (A_1 b_1 + C) \left( -y_2 + y_p - \frac{F_{21t}}{2} \right) b_2$$

$$\quad + \frac{1}{2} (A_{16}^4 A_3 - B_{16}^4) \left( x_3 - x_p - \frac{F_{21t}}{2} \right) b_1$$

$$\quad + A_{16}^4 (A_1 b_1 + A_2 b_2 + C) \left( x_3 - x_p - \frac{F_{21t}}{2} \right) b_1$$

$$\quad + \frac{1}{2} (A_{16}^4 A_4 - B_{16}^4) \left( y_4 - y_p - \frac{F_{21t}}{2} \right) b_2$$

$$\quad + A_{16}^4 (C + A_1 b_1 + A_2 b_2 + A_3 b_1)$$

$$\quad \times (y_4 - y_p - \frac{F_{21t}}{2} b_2$$

$$\quad E_{56} = A_{16}^2 \frac{F}{21r_2} b_2 + B_{16}^2 b_2 - A_{16}^4 \frac{F}{21r_4} b_2 - B_{16}^4 b_2$$

$$\quad E_{57} = -A_{16}^4 \frac{F}{21l_1} b_1 - B_{16}^4 b_1 + A_{16}^4 \frac{F}{21t_3} b_1 + B_{16}^4 b_1$$

$$\quad E_{58} = \left( A_{16}^4 \frac{F}{21l_1} + B_{16}^4 \right) \left( -x_1 + x_p - \frac{F_{21l}}{2} \right) b_1$$

$$\quad + \left( A_{16}^4 \frac{F}{21r_2} + B_{16}^4 \right) \left( -y_2 + y_p - \frac{F_{21t}}{2} \right) b_2$$

$$\quad + \left( A_{16}^4 \frac{F}{21r_3} + B_{16}^4 \right) \left( x_3 - x_p - \frac{F_{21t}}{2} \right) b_1$$

$$\quad + \left( A_{16}^4 \frac{F}{21t_4} + B_{16}^4 \right) \left( y_4 - y_p - \frac{F_{21t}}{2} \right) b_2$$
\[ E_{66} = A_{55}^1 b_1 + A_{66}^2 b_2 + A_{55}^3 b_1 + A_{66}^4 b_2 \]
\[ E_{67} = 0 \]  
(39p)

\[ E_{68} = -\frac{1}{2} A_{55}^1 b_1^2 + A_{66}^2 \left(-y_2 + y_p - \frac{F}{2t_2}\right) b_2 \]
\[ + \frac{1}{2} A_{55}^3 b_1^2 - A_{66}^4 \left(y_4 - y_p - \frac{F}{2t_4}\right) b_2 \]  
(39q)

\[ E_{77} = A_{66}^1 b_1 + A_{66}^2 b_2 + A_{55}^3 b_1 + A_{55}^4 b_2 \]  
(39r)

\[ E_{78} = -A_{66}^1 \left(x_1 + x_p - \frac{F}{2t_1}\right) b_1 \]
\[ + A_{66}^3 \left(x_3 - x_p - \frac{F}{2t_3}\right) b_1 \]  
(39s)

\[ E_{88} = A_{66}^1 \left(-x_1 + x_p - \frac{F}{2t_1}\right)^2 b_1 \]
\[ + \frac{1}{3} A_{55}^1 b_1^3 + A_{66}^2 \left(-y_2 + y_p - \frac{F}{2t_2}\right)^2 b_2 \]
\[ + \frac{1}{3} A_{55}^4 b_2^3 + A_{66}^3 \left(x_3 - x_p - \frac{F}{2t_3}\right)^2 b_1 \]
\[ + \frac{1}{3} A_{55}^3 b_1^3 + A_{66}^4 \left(y_4 - y_p - \frac{F}{2t_4}\right)^2 b_2 \]  
(39t)

where the St. Venant circuit shear flow \( F \), the constant \( C \) and other values of \( \tilde{E}_{ij} \) can be found in Ref. [13]

**References**


