A two variable refined plate theory for laminated composite plates

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A B S T R A C T

A two variable refined plate theory of laminated composite plates is developed in this paper. The theory accounts for parabolic distribution of the transverse shear strains, and satisfies the zero traction boundary conditions on the surfaces of the plate without using shear correction factor. Equations of motion are derived from the Hamilton’s principle. The closed-form solutions of antisymmetric cross-ply and angle-ply laminates are obtained using Navier solution. Numerical results of present theory are compared with three-dimensional elasticity solutions and results of the first-order and the other higher-order theories reported in the literature. It can be concluded that the proposed theory is accurate and simple in solving the static bending and buckling behaviors of laminated composite plates.

1. Introduction

Fiber reinforced composite are widely used in the aerospace, automotive, marine and other structural applications. In the past three decades, researches on laminated composite plates have received great attention, and a variety of laminated theories has been introduced. The classical laminate plate theory (CLPT), which neglects the transverse shear effects, provides reasonable results for thin plates. However, the CLPT underpredicts deflections and overpredicts frequencies as well as buckling loads with moderately thick plates. Many shear deformation theories account for transverse shear effects and overpredicts frequencies as well as buckling loads with moderately thick plates. Shear deformation theories include higher-order terms in Taylor’s expansions of the thickness coordinate, and they account for the transverse shear effects by the way of linear variation of in-plane displacements through the thickness. Since FSDT violates equilibrium conditions at the top and bottom faces of the plate, shear correction factors are required to rectify the unrealistic variation of the shear strain/stress through the thickness. In order to overcome the limitations of FSDT, higher-order shear deformation theories (HSDTs), which involve higher-order terms in Taylor’s expansions of the thickness coordinate, were developed by Librescu [3], Levinson [4], Bhimaraddi and Stevens [5], Reddy [6], Ren [7], Kant and Pandya [8], and Mohan et al. [9].

A good review of these theories for the analysis of laminated composite plates is available in Refs. [10–14]. A two variable refined plate theory (RPT) using only two unknown functions was developed by Shimpi [15] for isotropic plates, and was extended by Shimpi and Patel [16,17] for orthotropic plates. The most interesting feature of this theory is that it does not require shear correction factor, and has strong similarities with the classical plate theory in some aspects such as governing equation, boundary conditions and moment expressions.

The purpose of this paper is to develop the RPT for laminated composite plates. The present theory satisfies equilibrium conditions at the top and bottom faces of the plate without using shear correction factors. Governing equations are derived from the Hamilton’s principle. Navier solution is used to obtain the closed-form solutions for simply supported antisymmetric cross-ply and angle-ply laminates. To illustrate the accuracy of the present theory, the obtained results are compared with three-dimensional elasticity solutions and results of the first-order and the other higher-order theories.

2. Refined plate theory for laminated composite plates

2.1. Basic assumptions

Consider a rectangular plate of total thickness h composed of n orthotropic layers with the coordinate system as shown in Fig. 1. Assumptions of the RPT are as follows:

(i) The displacements are small in comparison with the plate thickness and, therefore, strains involved are infinitesimal.

(ii) The transverse displacement \(W\) includes three components of extension \(w_a\), bending \(w_b\), and shear \(w_c\). These components are functions of coordinates \(x, y, t\) only.

\[
W(x, y, z, t) = w_a(x, y, t) + w_b(x, y, t) + w_c(x, y, t)
\]  

(iii) The transverse normal stress \(\sigma_z\) is negligible in comparison with in-plane stresses \(\sigma_x\) and \(\sigma_y\).
placements in Eq. (4) are small in most cases. It can be neglected for a simpler version of the classical plate theory. The extension component of the transverse displacement is negligible small in most cases. It can be neglected for a simpler version of the classical plate theory.

2.2. Kinematics

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (1)–(3) as

\[ U(x, y, z, t) = u(x, y, t) + z \frac{\partial w_{0}}{\partial x} + \frac{2}{3} z^2 \left( \frac{\partial w_{0}}{\partial x} \right)^2 \]

\[ V(x, y, z, t) = v(x, y, t) + z \frac{\partial w_{0}}{\partial y} + \frac{2}{3} z^2 \left( \frac{\partial w_{0}}{\partial y} \right)^2 \]

\[ W(x, y, z, t) = w(x, y, t) + w_{0}(x, y, t) + \frac{2}{3} \frac{\partial w_{0}}{\partial z} \]

The extension component \( w_{0} \) of transverse displacement is negligible small in most cases. It can be neglected for a simpler version of the present theory named RPT1. The strains associated with the displacements in Eq. (4) are

\[ \varepsilon_{x} = \varepsilon^0_{x} + Z \kappa_{x} + f \kappa'_{x} \]

\[ \varepsilon_{y} = \varepsilon^0_{y} + Z \kappa_{y} + f \kappa'_{y} \]

\[ \gamma_{xy} = \gamma^0_{xy} + Z \kappa_{xy} + f \kappa'_{xy} \]

\[ \sigma_{x} = \sigma^0_{x} + 2 \frac{\partial w_{0}}{\partial x} \]

\[ \sigma_{y} = \sigma^0_{y} + 2 \frac{\partial w_{0}}{\partial y} \]

\[ \sigma_{xy} = \frac{\partial w_{0}}{\partial x} \]

\[ \tau_{xy} = \frac{\partial w_{0}}{\partial y} \]

\[ \nu_{x} = 0 \]

\[ q = 0 \]

2.3. Constitutive equations

Under the assumption that each layer possesses a plane of elastic symmetry parallel to the \( x-y \) plane, the constitutive equations for a layer can be written as

\[ \begin{bmatrix} \sigma_{x} \\ \sigma_{y} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{x} \\ \varepsilon_{y} \\ \gamma_{xy} \end{bmatrix} \]

where \( Q_{ij} \) are the plane stress-reduced stiffnesses, and are known in terms of the engineering constants in the material axes of the layer:

\[ Q_{ij} = \begin{bmatrix} E_{1} & -v_{12}E_{2} & 0 \\ -v_{12}E_{2} & E_{2} & 0 \\ 0 & 0 & G_{66} \end{bmatrix} \]

Since the laminate is made of several orthotropic layers with their material axes oriented arbitrarily with respect to the laminate coordinates, the constitutive equations of each layer must be transformed to the laminate coordinates \( (x, y, z) \). The stress-strain relations in the laminate coordinates of the \( k \)th layer are given as

\[ \begin{bmatrix} \sigma_{x}^{(k)} \\ \sigma_{y}^{(k)} \\ \sigma_{xy}^{(k)} \end{bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 \\ Q_{12}^{(k)} & Q_{22}^{(k)} & 0 \\ 0 & 0 & Q_{66}^{(k)} \end{bmatrix} \begin{bmatrix} \varepsilon_{x}^{(k)} \\ \varepsilon_{y}^{(k)} \\ \gamma_{xy}^{(k)} \end{bmatrix} \]

where \( Q_{ij}^{(k)} \) are the transformed material constants given as

\[ Q_{11}^{(k)} = Q_{11} \cos^{4} \theta + 2(Q_{12} + 2Q_{66}) \sin^{2} \theta \cos^{2} \theta + Q_{22} \sin^{4} \theta \]

\[ Q_{12}^{(k)} = (Q_{11} + Q_{22} - 4Q_{66}) \sin^{2} \theta \cos^{2} \theta + Q_{12} \sin^{4} \theta + Q_{22} \cos^{4} \theta \]

\[ Q_{16}^{(k)} = (Q_{11} - Q_{12} + 2Q_{66}) \sin \theta \cos \theta + (Q_{12} - Q_{12} + 2Q_{66}) \sin^{3} \theta \cos \theta \]

\[ Q_{26}^{(k)} = (Q_{11} - Q_{12} - 2Q_{66}) \sin^{3} \theta \cos \theta + (Q_{12} + Q_{12} + 2Q_{66}) \sin^{3} \theta \cos \theta \]

\[ Q_{66}^{(k)} = (Q_{11} + Q_{22} - 2Q_{66}) \sin^{2} \theta \cos^{2} \theta + Q_{12} (Q_{12} + 2Q_{66}) \sin^{4} \theta + Q_{22} \cos^{4} \theta \]

\[ Q_{44}^{(k)} = Q_{44} \cos^{2} \theta + Q_{55} \sin^{2} \theta \]

\[ Q_{45}^{(k)} = Q_{55} \cos \theta + Q_{44} \sin \theta \]

\[ Q_{55}^{(k)} = Q_{55} \cos^{2} \theta + Q_{44} \sin^{2} \theta \]
in which \( \theta \) is the angle between the global x-axis and the local x-axis of each lamina.

2.4. Equation of motions

The strain energy of the plate can be written as:

\[
U = \frac{1}{2} \int_V \sigma_{ij} \varepsilon_{ij} \, dV
\]

\[
= \frac{1}{2} \int_V (\sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{xy} \varepsilon_{xy} + \sigma_{zx} \varepsilon_{zx}) \, dV
\]

(11)

Substituting Eqs. (5) and (9) into Eq. (11) and integrating through the thickness of the plate, the strain energy of the plate can be rewritten as:

\[
U = \frac{1}{2} \int_{-h/2}^{h/2} \left[ \left( N_{xx}^0 \varepsilon_{xx} + N_{yy}^0 \varepsilon_{yy} + N_{xy}^0 \varepsilon_{xy} \right) + M_{xx}^0 \varepsilon_{xx}^1 + M_{yy}^0 \varepsilon_{yy}^1 + M_{xy}^0 \varepsilon_{xy}^1 \right] \, dx dy
\]

\[
+ \left( Q_{xz}^0 \varepsilon_{xz}^1 + Q_{yz}^0 \varepsilon_{yz}^1 + Q_{zx}^0 \varepsilon_{zx}^1 \right) \]  

(12)

where the stress resultants \( N, M, \) and \( Q \) are defined by:

\[
(N_{xx}, N_{yy}, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) \, dz = \sum_{k=1}^{N_z} \int_{z_k-1}^{z_k} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) \, dz
\]

\[
(M_{xx}^0, M_{yy}^0, M_{xy}^0) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) \, dz = \sum_{k=1}^{N_z} \int_{z_k-1}^{z_k} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) \, dz
\]

\[
(M_{xz}^0, M_{yz}^0, M_{zx}^0) = \int_{-h/2}^{h/2} (\sigma_{xz}, \sigma_{yz}, \sigma_{zx}) \, dz = \sum_{k=1}^{N_z} \int_{z_k-1}^{z_k} (\sigma_{xz}, \sigma_{yz}, \sigma_{zx}) \, dz
\]

(13)

Substituting Eq. (9) into Eq. (13) and integrating through the thickness of the plate, the stress resultants are given as:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix}
= \int_{-h/2}^{h/2}
\begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix}
\begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\frac{y}{h} \\
\frac{y}{h} \\
\frac{y}{h}
\end{bmatrix}
\]

\[
Q_{xz}^0
= \int_{-h/2}^{h/2}
\begin{bmatrix}
A_{44} & A_{45} & A_{46} \\
A_{45} & A_{55} & A_{56} \\
A_{46} & A_{56} & A_{66}
\end{bmatrix}
\begin{bmatrix}
\frac{y}{h} \\
\frac{y}{h} \\
\frac{y}{h}
\end{bmatrix}
\]

(14a)

motion appropriate to the displacement field and the constitutive equation. The principle can be stated in analytical form as:

\[
0 = \int_0^t \delta (U + V - T) \, dt
\]

(19)

where \( \delta \) indicates a variation with respect to \( x \) and \( y \).

Substituting Eqs. (12), (16) and (17) into Eq. (19) and integrating the equation by parts, collecting the coefficients of \( \delta u, \delta v, \)
\( \delta w_0, \delta w_b, \) and \( \delta w_a, \) the equations of motion for the laminate plate are obtained as follows:

\[
\begin{align*}
\delta u : \quad & \frac{\partial^2 N_{x}}{\partial x^2} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u} \\
\delta v : \quad & \frac{\partial^2 N_{y}}{\partial x^2} + \frac{\partial N_{yx}}{\partial y} = I_0 \ddot{v} \\
\delta w_b : \quad & \frac{\partial^2 M_{x}}{\partial x^2} + 2 \frac{\partial M_{xy}}{\partial x} + \frac{\partial^2 M_{y}}{\partial y^2} + q + N(w) \\
= & \left. I_0 (\dddot{w}_a + \dddot{w}_b + \dddot{w}_a) - I_2 \nabla^2 w_b \right|_{(20)} \\
\delta w_a : \quad & \frac{\partial^2 Q_{x}}{\partial x^2} + \frac{\partial Q_{yx}}{\partial y} + q + N(w) = I_0 (\dddot{w}_a + w_b + w_a)
\end{align*}
\]

where \( N(w) \) is defined by

\[
N(w) = N_{x} \frac{\partial^2 (w_a + w_b + w_a)}{\partial x^2} + N_{y} \frac{\partial^2 (w_a + w_b + w_a)}{\partial y^2} + 2 N_{xy} \frac{\partial^2 (w_a + w_b + w_a)}{\partial x \partial y} \tag{21}
\]

Eq. (20) can be expressed in terms of displacements \((u, v, w_b, w_a, w_a)\) by substituting for the stress resultants from Eq. (14). For homogeneous laminates, the equations of motion (20) take the form

\[
\begin{align*}
A_{11} & \frac{\partial^2 u}{\partial x^2} + 2 A_{16} \frac{\partial^2 u}{\partial y^2} + A_{66} \frac{\partial^2 u}{\partial y^2} + A_{66} \frac{\partial^2 v}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 v}{\partial x \partial y} + A_{26} \frac{\partial^2 v}{\partial y^2} \\
= & \left[ B_{11} \frac{\partial^2 w_b}{\partial x^2} + 3 B_{16} \frac{\partial^2 w_b}{\partial y^2} + (B_{12} + 2 B_{66}) \frac{\partial^2 w_b}{\partial x \partial y} + B_{26} \frac{\partial^2 w_a}{\partial y^2} \right] \tag{22a} \\
B_{11} & \frac{\partial^2 w_a}{\partial x^2} + 3 B_{16} \frac{\partial^2 w_a}{\partial y^2} + (B_{12} + 2 B_{66}) \frac{\partial^2 w_a}{\partial x \partial y} + B_{26} \frac{\partial^2 w_a}{\partial y^2} = I_0 \ddot{u} \\
A_{16} & \frac{\partial^2 u}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u}{\partial x \partial y} + A_{66} \frac{\partial^2 u}{\partial y^2} + A_{66} \frac{\partial^2 v}{\partial x^2} + 2 A_{26} \frac{\partial^2 v}{\partial x \partial y} + A_{26} \frac{\partial^2 v}{\partial y^2} \\
= & \left[ B_{11} \frac{\partial^2 w_b}{\partial x^2} + 3 B_{16} \frac{\partial^2 w_b}{\partial y^2} + (B_{12} + 2 B_{66}) \frac{\partial^2 w_b}{\partial x \partial y} + B_{26} \frac{\partial^2 w_b}{\partial y^2} \right] \tag{22b} \\
B_{11} & \frac{\partial^2 w_a}{\partial x^2} + 3 B_{16} \frac{\partial^2 w_a}{\partial y^2} + (B_{12} + 2 B_{66}) \frac{\partial^2 w_a}{\partial x \partial y} + B_{26} \frac{\partial^2 w_a}{\partial y^2} + B_{11} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + (B_{12} + 2 B_{66}) \frac{\partial^2 v}{\partial x \partial y} + B_{26} \frac{\partial^2 v}{\partial y^2} \\
= & \left[ D_{11} \frac{\partial^2 w_b}{\partial x^2} + 4 D_{16} \frac{\partial^2 w_b}{\partial y^2} + 2 (D_{12} + 2 D_{66}) \frac{\partial^2 w_b}{\partial x \partial y} \right] \tag{22c} \\
D_{11} & \frac{\partial^2 w_a}{\partial x^2} + 4 D_{16} \frac{\partial^2 w_a}{\partial y^2} + 2 (D_{12} + 2 D_{66}) \frac{\partial^2 w_a}{\partial x \partial y} + 4 D_{26} \frac{\partial^2 w_a}{\partial x \partial y} + D_{26} \frac{\partial^2 w_a}{\partial y^2} \\
= & \left[ 4 D_{11} \frac{\partial^2 w_a}{\partial x^2} + 4 D_{16} \frac{\partial^2 w_a}{\partial y^2} + 2 (D_{12} + 2 D_{66}) \frac{\partial^2 w_a}{\partial x \partial y} \right] \tag{24}
\end{align*}
\]

### 3. Analytical solutions for antisymmetric cross-ply laminates

The Navier solutions can be developed for rectangular laminates with two sets of simply supported boundary conditions (Fig. 2). For antisymmetric cross-ply laminates, the following plate stiffnesses are identically zero:

\[
A_{16} = A_{26} = D_{16} = D_{26} = D_{16} = D_{26} = F_{16} = F_{26} = H_{16} = H_{26} = H_{16} = H_{26} = 0 \tag{23}
\]

The following SS-1 boundary conditions for antisymmetric cross-ply laminates can be written as

\[
v(0, y) = w_a(0, y) = w_b(0, y) = w_a(0, y) = \frac{\partial w_a}{\partial y}(0, y) = \frac{\partial w_b}{\partial y}(0, y) = 0 \\
v(a, y) = w_a(a, y) = w_a(a, y) = \frac{\partial w_a}{\partial y}(a, y) = \frac{\partial w_a}{\partial y}(a, y) = 0 \\
N_s(0, x) = M_s(0, y) = M_s(0, y) = N_s(0, x) = M_s(0, y) = M_s(0, y) = 0 \tag{24}
\]

\[
u(x, 0) = w_2(x, 0) = w_b(x, 0) = w_a(x, 0) = \frac{\partial w_a}{\partial x}(x, 0) = \frac{\partial w_a}{\partial x}(x, 0) = 0 \\
u(x, b) = w_2(x, b) = w_b(x, b) = w_a(x, b) = \frac{\partial w_a}{\partial x}(x, b) = \frac{\partial w_a}{\partial x}(x, b) = 0 \\
N_s(x, 0) = M_s(0, x) = M_s(0, y) = N_s(x, 0) = M_s(x, b) = M_s(x, b) = 0 \tag{24}
\]
The boundary conditions in Eq. (24) are satisfied by the following expansions

\[
\begin{align*}
    u &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \cos \alpha x \sin b y \\
    v &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin \alpha x \cos b y \\
    w_b &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{bmn} \sin \alpha x \sin b y \\
    w_s &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{smn} \sin \alpha x \sin b y \\
    w_a &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{amn} \sin \alpha x \sin b y
\end{align*}
\]

(25)

where \( \alpha = \pi / a \), \( \beta = \pi / b \), and \((U_{mn}, V_{mn}, W_{bmn}, W_{smn}, W_{amn})\) are coefficients.

The applied transverse load \( q \) is also expanded in the double-Fourier series as

\[
q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \sin \alpha x \sin b y
\]

(26)

Substituting Eqs. (23), (25) and (26) into Eq. (22), the Naviol solution of antisymmetric cross-ply laminates can be determined from equations

\[
\begin{bmatrix}
    s_{11} & s_{12} & s_{13} & s_{14} & 0 \\
    s_{12} & s_{22} & s_{23} & s_{24} & 0 \\
    s_{13} & s_{23} & s_{33} + k & s_{34} + k & k \\
    s_{14} & s_{24} & s_{34} + k & s_{44} + k & s_{45} + k \\
    0 & 0 & 0 & k & s_{55} + k + k
\end{bmatrix}
\begin{bmatrix}
    U_{mn} \\
    V_{mn} \\
    W_{bmn} \\
    W_{smn} \\
    W_{amn}
\end{bmatrix}
\begin{bmatrix}
    m_{11} & 0 & 0 & 0 & 0 \\
    0 & m_{22} & 0 & 0 & 0 \\
    0 & 0 & m_{33} & m_{34} & m_{35} \\
    0 & 0 & m_{43} & m_{44} & m_{45} \\
    0 & 0 & m_{54} & m_{55} & m_{55}
\end{bmatrix}
\begin{bmatrix}
    U_{mm} \\
    V_{mm} \\
    W_{bmm} \\
    W_{smm} \\
    W_{amm}
\end{bmatrix}
\begin{bmatrix}
    0 \\
    0 \\
    Q_{mn}
\end{bmatrix}
\]

(27)

where

\[
\begin{align*}
    s_{11} &= A_{11} x^2 + A_{66} \beta^2, \quad s_{12} = \alpha \beta (A_{12} + A_{66}) \\
    s_{13} &= -B_{11} x^2, \quad s_{14} = -B_{66} \beta^2 \\
    s_{22} &= A_{66} x^2 + A_{12} \beta^2, \quad s_{23} = B_{11} x^2, \quad s_{24} = B_{11} \beta^2 \\
    s_{33} &= D_{11} x^4 + 2(D_{12} + 2 \nu_{66}) x^2 \beta^2 + D_{22} \beta^2 \\
    s_{34} &= D_{11} x^4 + 2(D_{12} + 2 \nu_{66}) x^2 \beta^2 + D_{22} \beta^2 \\
    s_{44} &= H_{11} x^2 + 2(H_{12} + 2 \nu_{66}) x^2 \beta^2 + H_{22} \beta^2 + A_{12} x^2 + A_{24} \beta^2 \\
    s_{45} &= A_{12} x^2 + A_{24} \beta^2, \quad s_{55} = A_{55} x^2 + A_{45} \beta^2 \\
    m_{11} &= m_{22} = m_{33} = m_{34} = m_{44} = m_{45} = m_{55} = I_0 \\
    m_{13} &= I_0 + I_2 (x^2 + \beta^2), \quad m_{44} = I_0 + I_2 84 (x^2 + \beta^2) \\
    k &= N_0^0 x^2 + N_0^0 \beta^2
\end{align*}
\]

(28)

4. Analytical solutions for antisymmetric angle-ply laminates

For antisymmetric angle-ply laminates, the following plate stiffnesses are identically zero:

\[
\begin{align*}
    A_{16} &= A_{26} = D_{16} = D_{26} = D_{16}' = D_{26}' = 0 \\
    F_{16} &= F_{26} = H_{16} = H_{26} = H_{16}' = H_{26}' = 0 \\
    B_{11} &= B_{12} = B_{22} = B_{66} = B_{11}' = B_{12}' \\
    B_{12}' &= B_{16}' = E_{11} = E_{12} = E_{22} = E_{66} = 0 \\
    A_{45} &= A_{35}' = A_{45}' = D_{45} = F_{45} = 0
\end{align*}
\]

(29)

The following SS-2 boundary conditions (Fig. 2) for antisymmetric angle-ply laminates can be written as

\[
\begin{align*}
    u(0,y) &= w_a(0,y) = w_b(0,y) = w_s(0,y) = \frac{\partial w_a}{\partial y}(0,y) = \frac{\partial w_b}{\partial y}(0,y) \\
    &= \frac{\partial w_s}{\partial y}(0,y) = 0 \\
    u(a,y) &= w_a(a,y) = w_b(a,y) = w_s(a,y) = \frac{\partial w_a}{\partial y}(a,y) = \frac{\partial w_b}{\partial y}(a,y) = \frac{\partial w_s}{\partial y}(a,y) = 0
\end{align*}
\]
In this section, various numerical examples are described and discussed for verifying the accuracy of the RPT in predicting the static bending and buckling behaviors of simply supported antisymmetric cross-ply and angle-ply laminates. For the verification purpose, the results obtained by RPT are compared with those of the CLPT, FSDT, HSDT, and exact solution of three-dimensional elasticity. In order to investigate the efficiency of the present theory, a simpler version of proposed theory (RPT1) is also developed by omitting the extension component $w_a$ of transverse displacement. The description of various displacement models is given in Table 1. In all examples, a shear correction factor of 5/6 is used for FSDT. The following lamina properties are used:

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_1$</th>
<th>$G_{12}$</th>
<th>$G_{23}$</th>
<th>$v_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25E_2</td>
<td>0.5E_2</td>
<td>0.2E_2</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>40E_2</td>
<td>0.5E_2</td>
<td>0.6E_2</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>40E_2</td>
<td>0.6E_2</td>
<td>0.8E_2</td>
<td>0.25</td>
</tr>
</tbody>
</table>

For convenience, the following nondimensionalizations are used in presenting the numerical results in graphical and tabular forms:

$$ \bar{w} = 100w(a/2, b/2) \left( \frac{E_1 h^3}{qa^4} \right) $$

$$ \bar{N} = N_{mn} \left( \frac{a^2}{E_1 h^2} \right) $$

### 5.1. Bending

The static bending solution can be obtained from Eq. (27) by setting the time derivative terms and in-plane forces to zero:

$$ \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & 0 & U_{mn} \\ s_{12} & s_{22} & s_{23} & s_{24} & 0 & V_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} $$

$$ \begin{bmatrix} s_{13} & s_{23} & s_{33} & s_{34} & 0 & W_{bmn} \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} & W_{umn} \end{bmatrix} = \begin{bmatrix} Q_{mn} \\ 0 \end{bmatrix} $$

For the case of RPT1 (Table 1), the static bending solution of can be simplified as

$$ \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & 0 & U_{mn} \\ s_{12} & s_{22} & s_{23} & s_{24} & 0 & V_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} $$

$$ \begin{bmatrix} s_{13} & s_{23} & s_{33} & s_{34} & 0 & W_{bmn} \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} & W_{umn} \end{bmatrix} = \begin{bmatrix} Q_{mn} \\ 0 \end{bmatrix} $$

### Example 1

A simply supported two-layer antisymmetric cross-ply (0/90) square laminate under sinusoidal transverse load is considered. Material set 1 is used. The numerical results and corresponding errors with respect to three-dimensional elasticity solution of nondimensionalized deflection are given in Table 2. From the results, it can be seen that the results obtained using HSDT and RPT1 (a simpler version of present theory) are identical.

### Table 1

<table>
<thead>
<tr>
<th>Model</th>
<th>Theory</th>
<th>Unknown function</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLPT</td>
<td>Classical laminate plate theory</td>
<td>3</td>
</tr>
<tr>
<td>FSDT</td>
<td>First-order shear deformation theory (Whitney and Pagano [22])</td>
<td>5</td>
</tr>
<tr>
<td>HSDT</td>
<td>Higher-order shear deformation theory (Reddy [6])</td>
<td>5</td>
</tr>
<tr>
<td>Ren</td>
<td>Higher-order shear deformation theory (Ren [20])</td>
<td>7</td>
</tr>
<tr>
<td>RPT</td>
<td>Refined plate theory without $w_a$ (Present)</td>
<td>4</td>
</tr>
<tr>
<td>RPT2</td>
<td>Refined plate theory with $w_a$ (Present)</td>
<td>5</td>
</tr>
</tbody>
</table>
In general, the present theory (RPT2) gives more accurate results in predicting deflections when compared to HSDT. Compared to the elasticity solution (Pagano [19]), RPT2 underpredicts deflection by 2.52% while HSDT underpredicts deflection by 3.57% for $a/h$ ratio equal to 5. Fig. 3 shows the variation of nondimensionalized deflection with respect to different side-to-thickness ratios using all models.

Example 2. A simply supported two-layer antisymmetric angle-ply $(45/-45)$ laminate under sinusoidal transverse load is considered. Material set 2 is used. The numerical results of nondimensionalized deflection for the square and rectangular plates are shown in Table 3. In the case of thick plates, there is a considerable difference exists between the results obtained using the various models and the values reported by Ren [20]. For $a/h$ ratio equal to 4, the deflections predicted by FSDT, HSDT, and RPT2 are

<table>
<thead>
<tr>
<th>$a/h$</th>
<th>Source</th>
<th>$w$</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Exact</td>
<td>4.9362</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>4.5619</td>
<td>–7.58</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>5.4103</td>
<td>9.60</td>
</tr>
<tr>
<td></td>
<td>RPT1</td>
<td>4.5619</td>
<td>–7.58</td>
</tr>
<tr>
<td></td>
<td>RPT2</td>
<td>4.8677</td>
<td>–1.39</td>
</tr>
<tr>
<td>5</td>
<td>Exact</td>
<td>1.7287</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>1.6670</td>
<td>–3.57</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>1.7627</td>
<td>1.97</td>
</tr>
<tr>
<td></td>
<td>RPT1</td>
<td>1.6670</td>
<td>–3.57</td>
</tr>
<tr>
<td></td>
<td>RPT2</td>
<td>1.6852</td>
<td>–2.52</td>
</tr>
<tr>
<td>10</td>
<td>Exact</td>
<td>1.2318</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>1.2161</td>
<td>–1.27</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>1.2416</td>
<td>0.80</td>
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<tr>
<td></td>
<td>RPT1</td>
<td>1.2161</td>
<td>–1.27</td>
</tr>
<tr>
<td></td>
<td>RPT2</td>
<td>1.2197</td>
<td>–0.98</td>
</tr>
<tr>
<td>20</td>
<td>Exact</td>
<td>1.1060</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>1.1018</td>
<td>–0.38</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>1.1113</td>
<td>0.48</td>
</tr>
<tr>
<td></td>
<td>RPT1</td>
<td>1.1018</td>
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</tr>
<tr>
<td></td>
<td>RPT2</td>
<td>1.1027</td>
<td>–0.30</td>
</tr>
<tr>
<td>100</td>
<td>Exact</td>
<td>1.0742</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>1.0553</td>
<td>–0.85</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>1.0553</td>
<td>–0.83</td>
</tr>
<tr>
<td></td>
<td>RPT1</td>
<td>1.0551</td>
<td>–0.85</td>
</tr>
<tr>
<td></td>
<td>RPT2</td>
<td>1.0551</td>
<td>–0.85</td>
</tr>
<tr>
<td></td>
<td>CLPT</td>
<td>1.0636</td>
<td>–0.99</td>
</tr>
</tbody>
</table>

Fig. 3. The effect of side-to-thickness ratio on nondimensionalized deflection of simply supported two-layer $(0/90)$ square laminates under sinusoidal transverse load.
20.01%, 29.49%, and 26.52% lower for a square plate and 14.72%, 20.41%, and 20.55% lower for a rectangular plate as compared to the values obtained by Ren [20]. The results computed using all the five models are in good agreement with those reported by Ren [20] for thin plates (a/h = 100). The nondimensionalized deflections of two-layer (45/−45) square laminates under sinusoidal transverse load are presented in Fig. 4 for various ratio of modulus \( E_I/E_2 \) (\( C_{11} = 0.5E_2, \ C_{22} = 0.6E_2, \ y_{12} = 0.25, a/h = 10 \)).

5.2. Buckling

For buckling analysis, the applied loads are assumed to be in-plane forces

\[ N_x^0 = -N_o, \quad N_y^0 = \gamma N_o, \quad \gamma = \frac{N_0^2}{N_o^2}, \quad N_y^0 = 0 \]  

(36)

The buckling solution can be obtained from Eq. (27) by setting the time derivative terms and transverse forces to zero:

\[
\begin{bmatrix}
    s_{11} & s_{12} & s_{13} & s_{14} & 0 \\
    s_{12} & s_{22} & s_{23} & s_{24} & 0 \\
    s_{13} & s_{23} & s_{33} - N_0(x^2 + \gamma\beta^2) & s_{34} & -N_0(x^2 + \gamma\beta^2) & -N_0(x^2 + \gamma\beta^2) \\
    s_{14} & s_{24} & s_{34} - N_0(x^2 + \gamma\beta^2) & s_{44} - N_0(x^2 + \gamma\beta^2) & s_{45} - N_0(x^2 + \gamma\beta^2) & s_{55} - N_0(x^2 + \gamma\beta^2) \\
    0 & 0 & -N_0(x^2 + \gamma\beta^2) & s_{45} - N_0(x^2 + \gamma\beta^2) & s_{55} - N_0(x^2 + \gamma\beta^2) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    U_{mn} \\
    V_{mn} \\
    W_{bmn} \\
    W_{abmn} \\
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0 \\
\end{bmatrix}
\]

(37)

Following the condensation of variables procedure to eliminate the in-plane displacements \( U_{mn} \) and \( V_{mn} \), the following system is obtained:

\[
\begin{bmatrix}
    s_{33} - N_0(x^2 + \gamma\beta^2) & s_{34} - N_0(x^2 + \gamma\beta^2) & -N_0(x^2 + \gamma\beta^2) \\
    s_{34} - N_0(x^2 + \gamma\beta^2) & s_{44} - N_0(x^2 + \gamma\beta^2) & s_{55} - N_0(x^2 + \gamma\beta^2) \\
    -N_0(x^2 + \gamma\beta^2) & s_{45} - N_0(x^2 + \gamma\beta^2) & s_{55} - N_0(x^2 + \gamma\beta^2) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    W_{bmn} \\
    W_{abmn} \\
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0 \\
\end{bmatrix}
\]

(38)

where

\[
\begin{align*}
    s_{33} &= s_{33} - \frac{b_1}{b_0} - \frac{b_2}{b_0}, \quad s_{34} = s_{34} - \frac{b_1}{b_0} - \frac{b_2}{b_0} \\
    s_{43} &= s_{34} - \frac{b_3}{b_0} - \frac{b_4}{b_0}, \quad s_{44} = s_{44} - \frac{b_1}{b_0} - \frac{b_2}{b_0} \\
    b_0 &= s_{11}s_{22} - s_{12}^2, \quad b_1 = s_{11}s_{22} - s_{12}s_{23}, \quad b_2 = s_{11}s_{22} - s_{12}s_{13} \\
    b_3 &= s_{14}s_{22} - s_{12}s_{24}, \quad b_4 = s_{11}s_{24} - s_{12}s_{14}
\end{align*}
\]

(39)

For nontrivial solution, the determinant of the coefficient matrix in Eq. (38) must be zero. This gives the following expression for buckling load:

\[
N_0 = \frac{1}{2x^2 + \gamma\beta^2} \times \left[ \frac{(s_{33}s_{44} - s_{34}s_{43})s_{55} - s_{33}s_{45}^2}{s_{33}(s_{44} + s_{55} - 2s_{45}) + (s_{34} + s_{43})(s_{45} - s_{55}) + s_{34}s_{55} - s_{35}^2 - s_{34}s_{43}} \right]
\]

(40a)

For the case of RPT1 (Table 1), the buckling load can be simplified as

\[
N_0 = \frac{1}{2x^2 + \gamma\beta^2} \times \frac{s_{33}s_{44} - s_{34}s_{43}}{s_{33} + s_{44} - s_{34} - s_{43}}
\]

(40b)

Example 3. A simply supported antisymmetric cross-ply (0/90)\( _n \) (n = 2, 3, 5) square laminate subjected to uniaxial compressive load on sides \( x = 0, a \) is considered. Material set 3 is used. Table 4 shows a comparison between the results obtained using the various models and the three-dimensional elasticity solutions given by Noor [21]. The results clearly indicate that the present theory (RPT2) gives more accurate results in predicting the buckling loads when compared to HDST, and results obtained using HDST and RPT1 (a simpler version of present theory) are identical. Compared to the three-dimensional elasticity solution, the buckling loads predicted by RPT2, HDST, and FSDT are 6.06%, 6.11%, and 7.17%, respectively, for four-layer antisymmetric cross-ply (0/90/0/90) square laminates. The effect of side-to-thickness ratio on buckling load of simply supported four-layer (0/90/0/90) square laminates is also presented in Fig. 5.

Example 4. A simply supported two-layer antisymmetric angle-ply (0/−0)\( _n \) square laminate subjected to uniaxial compressive load on sides \( x = 0, a \) is considered. Material set 3 is used. The numerical values of nondimensionalized buckling load are given in Table 5. The results are compared with the values reported by Ren [20]. For all values of side-to-thickness ratio and fiber orientation, the buckling loads predicted by the RPT1 and HDST are almost identical. For \( a/h \) ratio equal to 4 and the fiber orientation equal to 30°, the buckling load values predicted by FSDT, HDST, and RPT1 are

![Fig. 5. The effect of side-to-thickness ratio on nondimensionalized uniaxial buckling load of simply supported four-layer (0/90/0/90) square laminates.](source: URL)

<table>
<thead>
<tr>
<th>Source</th>
<th>( a/h )</th>
<th>( \theta = 30^\circ )</th>
<th>( \theta = 45^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>HDST</td>
<td>9.3391</td>
<td>8.2377</td>
<td></td>
</tr>
<tr>
<td>FSDT</td>
<td>7.5450</td>
<td>6.7858</td>
<td></td>
</tr>
<tr>
<td>RPT1</td>
<td>9.3518</td>
<td>8.3963</td>
<td></td>
</tr>
<tr>
<td>RPT2</td>
<td>8.5446</td>
<td>7.7898</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Ren [20]</td>
<td>15.7517</td>
<td>16.4558</td>
</tr>
<tr>
<td>HDST</td>
<td>17.1269</td>
<td>18.1544</td>
<td></td>
</tr>
<tr>
<td>FSDT</td>
<td>16.6132</td>
<td>17.5522</td>
<td></td>
</tr>
<tr>
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<td>RPT2</td>
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<td>18.0031</td>
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<tr>
<td>CLPT</td>
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</table>

Table 5: Nondimensionalized uniaxial buckling load of simply supported two-layer (0/−0)\( _n \)\( \) square laminates.
behaviors of antisymmetric cross-ply and angle-ply laminated composite plates.

Acknowledgements

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References