Advanced form-finding of tensegrity structures

Hoang Chi Tran, JaeHong Lee

Department of Architectural Engineering, Sejong University, 98 Kunja Dong, Kwangjin Ku, Seoul 143-747, Republic of Korea

A R T I C L E   I N F O

Article history:
Received 22 May 2009
Accepted 28 October 2009
Available online 30 November 2009

Keywords:
Tensegrity structure
Single value decomposition
Form-finding
Eigenvalue problem

A B S T R A C T

A numerical method is presented for form-finding of tensegrity structures. The topology and the types of members are the only information that requires in this form-finding process. The eigenvalue decomposition of the force density matrix and the single value decomposition of the equilibrium matrix are performed iteratively to find the feasible sets of nodal coordinates and force densities which satisfy the minimum required rank deficiencies of the force density and equilibrium matrices, respectively. Based on numerical examples it is found that the proposed method is very efficient and robust in searching self-equilibrium configurations of tensegrity structures.

© 2009 Elsevier Ltd. All rights reserved.

1. Introduction

The tensegrity structures first proposed by Fuller [1] have been developed in recent years due to their innovative forms, lightweight and deployability. They belong to a class of free-standing pre-stressed pin-jointed cable–strut system where contacts are allowed among the struts [2]. Their classification is presented as Class 1 (where bars do not touch) and Class 2 (where bars do connect to each other at a pivot) [3]. As a pioneering work of form-finding, so-called force density method was proposed by Schek [4] for form-finding of tensile structures. Motro et al. [5] presented the dynamic relaxation which has been reliably applied to tensile structures [6] and many other non-linear problems. Vassart and Motro [7] employed the force density method in symbolic form for searching new configurations. Recently, Masic et al. [8], Zhang and Ohsaki [9] and Estrada et al. [10] developed new numerical methods using a force density formulation. Micheletti et al. [11] used a marching procedure for finding stable placements of a given tensegrity, and Zhang et al. [12] employed a refined dynamic relaxation procedure for form-finding of nonregular tensegrity systems. Most recently, Rieffel et al. [13] introduced an evolutionary algorithm for producing large irregular tensegrity structures. Tibert and Pellegrino [14] presented a review paper for the existing methods for form-finding of tensegrity structures. The most recent review for this problem can be found in Juan and Tur [15].

In most available form-finding methods, the symmetry of structure and/or some of member lengths must be assumed known in advance. Moreover, element force density coefficients are solved in symbolic form. For example, (i) symbolic properties are employed in a group theory so as to simplify form-finding problem as presented in [8]; (ii) a number of member lengths are specified at the start in a dynamic relaxation procedure and non-linear programming, as presented in [5,6,14]; however, these information may not always be available or easy to define at the beginning; and (iii) force density coefficients are considered as symbolic variables which cannot be applied for structures with large number of members [14].

In this paper, a numerical method is presented for form-finding of tensegrity structures. Both Classes 1 and 2 of tensegrity structures are considered. Tensegrities satisfying either stability (i.e., the tangent stiffness matrix is positive definite) or super stability (i.e., the geometrical stiffness matrix is positive definite) can be obtained by present form-finding procedure in a few of remarkable iterations, which is more efficient and versatile than other available methods only dealing with super stable tensegrities (e.g., [9,10]) that are more restrictive in the real mechanical structures. The topology and the types of members, i.e. either compression or tension, are the only information that requires in this form-finding process. In other words, any initial nodal coordinate, symmetric properties, and member lengths are not necessary for the present form-finding. The force density matrix is derived from an incidence matrix and an initial set of force densities assigned from prototypes, while the equilibrium matrix is defined by the incidence matrix and nodal coordinates. The eigenvalue decomposition of the force density matrix and the single value decomposition of the equilibrium matrix are performed iteratively to find the feasible sets of nodal coordinates and force densities which satisfy the minimum required rank deficiencies of the force density and equilibrium matrices, respectively. The evaluation of the eigenvalues of tangent stiffness is also included for checking the stability of the tensegrity structures.
2. Formulation of self-equilibrium equations

2.1. Fundamental assumptions

In this study, the following assumptions are made in tensegrity structures:

- The topology of the structure in terms of nodal connectivity is known.
- Members are connected by pin joints.
- No external load is applied and the self-weight of the structure is neglected during the form-finding procedure.
- There are no dissipative forces acting on the system.

2.2. Force density method for the self-equilibrium and free-standing tensegrity structures

For a d-dimensional tensegrity structure with b members, n free nodes and ni fixed nodes (supports), its topology can be expressed by a connectivity matrix \( C \in \mathbb{R}^{b \times n} \) as discussed in \[4,16,19\]. Suppose member \( k \) connects nodes \( i \) and \( j \) (\( i < j \)), then the \( i \)th and \( j \)th elements of the \( k \)th row of \( C \) are set to 1 and \(-1\), respectively, as follows:

\[
C_{ik} = \begin{cases} 
1 & \text{for } p = i, \\
-1 & \text{for } p = j, \\
0 & \text{otherwise}. 
\end{cases} 
\tag{1}
\]

If the free nodes are numbered first, then to the fixed nodes, \( C \) can be divided into two parts as

\[
C = [C_f C'_f], 
\tag{2}
\]

where \( C_f \in \mathbb{R}^{b \times n} \) and \( C'_f \in \mathbb{R}^{n \times n} \) describe the connectivities of the members to the free and fixed nodes, respectively. Let \( x, y, z \in \mathbb{R}^n \) and \( x_i, y_j, z_j \in \mathbb{R}^n \) denote the nodal coordinate vectors of the free and fixed nodes, respectively, in x, y and z directions. For a simple two-dimensional two-strut tensegrity structure as shown in Fig. 1, which consists of six members \((b = 6, \text{four cables and two struts})\) and four nodes, the connectivity matrix \( C \in \mathbb{R}^{6 \times 4} \) is given in Table 1.

The equilibrium equations in each direction of a general pin-jointed structure given by \[4\] can be stated as

\[
C^t_q Q C x = p_x, 
\tag{3a}
\]

\[
C^t_q Q y = C^t_q Q C y = p_y, 
\tag{3b}
\]

\[
C^t_q Q z = C^t_q Q C z = p_z, 
\tag{3c}
\]

where \( p_x, p_y, \) and \( p_z \in \mathbb{R}^n \) are the vectors of external loads applied at the free nodes in x, y and z directions, respectively. The symbol, \( (\cdot)^t \), denotes the transpose of a matrix or vector. And \( Q \in \mathbb{R}^{n \times n} \) is diagonal square matrix, calculated by

\[
Q = \text{diag}(q), 
\tag{4}
\]

where \( q \in \mathbb{R}^n \) suggested in \[4\] is the force density vector, defined by

\[
q = [q_1, q_2, \ldots, q_b]^T, 
\tag{5}
\]

in which each component of this vector is the force \( q_i \) to length \( l_i \) ratio \( q_i/l_i \), known as force density or self-stressed coefficient in \[7\]. Without external loading, Eq. (3) can be rewritten neglecting the self-weight of the structure as

\[
D_x = -D_y x, 
\tag{6a}
\]

\[
D_y = -D_y y, 
\tag{6b}
\]

\[
D_z = -D_z z, 
\tag{6c}
\]

where matrices \( D \in \mathbb{R}^{n \times n} \) and \( D_f \in \mathbb{R}^{n \times n} \) are, respectively, given by

\[
D = C^t_q Q C, 
\tag{7a}
\]

\[
D_f = C^t_q Q C_f. 
\tag{7b}
\]

or by

\[
D = C^t_q \text{diag}(q) C, 
\tag{8a}
\]

\[
D_f = C^t_q \text{diag}(q) C_f. 
\tag{8b}
\]

It is noted that when external load and self-weight are ignored, a tensegrity system does not require any fixed node, and the tensegrity geometry is defined by the relative position of the nodes, and the system can be considered as free-standing, forming a rigid-body free in space \[7,16\]. In this context, Eq. (7b) and (8b) vanish, and Eq. (6) becomes:

\[
D_x = 0, 
\tag{9a}
\]

\[
D_y = 0, 
\tag{9b}
\]

\[
D_z = 0, 
\tag{9c}
\]

where \( D \) known as force density matrix \[14,10\] or stress matrix \[17–19\] can be written directly by \[7,16\] without using Eq. (7) or (8) as follows:

\[
D_{(ij)} = \begin{cases} 
-q_k & \text{if nodes } i \text{ and } j \text{ are connected by member } k, \\
0 & \text{otherwise} 
\end{cases} 
\tag{10}
\]

in which \( \Omega \) denotes the set of members connected to node \( i \). For example, for the two-dimensional two-strut tensegrity structure in Fig. 1, \( D \) can be written explicitly from Eq. (10) as

\[
D = \begin{bmatrix} 
q_1 + q_4 + q_5 & -q_1 & -q_5 & -q_4 \\
-q_1 & q_1 + q_2 + q_6 & -q_2 & -q_6 \\
-q_5 & -q_2 & q_2 + q_4 + q_5 & -q_3 \\
-q_4 & -q_6 & -q_3 & q_3 + q_4 + q_6
\end{bmatrix}, 
\tag{11}
\]

From Eq. (10), it is obvious that \( D \) is always square, symmetric and singular with a nullity of at least 1 since the sum of all components

![Fig. 1. A two-dimensional two-strut tensegrity structure. The thick and thin lines represent the struts and cables, respectively.](image-url)
in any row or column is zero for any tensegrity structure. Different from the matrix $D$ of the cable net, which is always positive definite\cite{4}, $D$ of the tensegrity structure is semi-definite due to the existence of struts as compression members, with $q_k < 0$. Consequently, it cannot be invertible. For simplicity, Eq. (9) can be reorganized as

$$D[x \ y \ z] = [0 \ \ 0 \ \ 0].$$  \hspace{1cm} (12)

However, by substituting Eq. (8) into Eq. (9) the equilibrium equations of the tensegrity structure can be expressed as

$$C^d dia (q)Cx = 0.$$  \hspace{1cm} (13a)

$$C^d dia (q)Cy = 0.$$  \hspace{1cm} (13b)

$$C^d dia (q)Cz = 0.$$  \hspace{1cm} (13c)

Eq. (13) can be reorganized as

$$Aq = 0.$$  \hspace{1cm} (14)

where $A \in \mathbb{R}^{dn \times b}$ is known as the equilibrium matrix in \cite{16}, defined by

$$A = \begin{pmatrix} C^d dia (Cx) \\ C^d dia (Cy) \\ C^d dia (Cz) \end{pmatrix}.$$  \hspace{1cm} (15)

Eq. (12) presents the relation between force densities and nodal coordinates, while Eq. (14) shows the relation between projected lengths in $x$, $y$ and $z$ directions, respectively and force densities. Both Eqs. (12) and (14) are linear homogeneous systems of self-equilibrium equations with respect to nodal coordinates and force densities, respectively.

3. Requirement on rank deficiency conditions of force density and equilibrium matrices

Let $q$ be the vector of force density and $C$ be the incidence matrix of a $d$-dimensional tensegrity structure in self-equilibrium. It is well known that the set of all solutions to the homogeneous system of Eq. (12) is the null space of $D$. The dimension of this null space or rank deficiency of $D$ is defined as

$$n_D = n - r_D,$$  \hspace{1cm} (16)

where $r_D = \text{rank}(D)$. It is obvious that vector $I_d = \{1, 1, \ldots, 1\}^T \in \mathbb{R}^{n_d}$ is a solution of Eq. (12) since the sum of the elements of a row or a column of $D$ is always equal to zero. The most important rank deficiency condition related to semi-definite matrix $D$ of Eq. (12) is defined by

$$n_D \geq d + 1.$$  \hspace{1cm} (17)

This condition forces Eq. (12) to yield at least $d$ useful particular solutions \cite{20} which exclude the above vector $I_d$, due to degenerating geometry of tensegrity structure \cite{14,9}. These $d$ particular solutions form a vector space basis for generating a $d$-dimensional tensegrity structure. Therefore, the minimum rank deficiency or nullity of $D$ must be $(d + 1)$ for configuration of any tensegrity structure embedding into $\mathbb{R}^d$, which is equivalent to the maximum rank condition of $D$ proposed by \cite{17,18,19} as follows:

$$\text{max}(r_D) = n - (d + 1).$$  \hspace{1cm} (18)

Similarly, the set of all solutions to the homogeneous system of Eq. (14) lies in the null space of $A$. Let $n_A$ denote dimension of null space of the equilibrium matrix $A$ which is computed by

$$n_A = b - r_A,$$  \hspace{1cm} (19)

where $r_A = \text{rank}(A)$. The second rank deficiency condition which ensures the existence of at least one state of self-stress can be stated as

$$s = n_A \geq 1.$$  \hspace{1cm} (20)

where $s$ is known as the number of independent states of self-stress, while the number of infinitesimal mechanisms is computed by $m = dn - r_A$, as presented in \cite{21,22}. It is clear that Eq. (20) allows Eq. (14) to create at least one useful particular solution \cite{20}.

In short, based on these two rank deficiency conditions, Eqs. (17) and (20), the proposed form-finding procedure searches for self-equilibrium configurations that permit the existence of at least one state of self-stress in the structure. It should be noted that these two are necessary but not sufficient conditions which have to be satisfied for any $d$-dimensional tensegrity structure to be in a self-equilibrium state \cite{17,14,16}. The sufficient conditions for tensegrity or self-stressed pin-jointed structures can be found in \cite{24,23}.

4. Form-finding process

The proposed form-finding procedure only needs to know the topology of structure in terms of the incidence matrix $C$, and type of each member, i.e. either cable or strut which is under tension or compression, respectively. Based on element type, the initial force density coefficients of cables (tension) are automatically assigned as $+1$ while those of struts (compression) as $-1$, respectively, as follows:

$$q^i = \begin{pmatrix} +1 +1 \cdots +1 \\ -1 -1 \cdots -1 \end{pmatrix}.$$  \hspace{1cm} (21)

Subsequently, the force density matrix $D$ is calculated from $q^i$ by Eq. (8). After that, the nodal coordinates are selected from the eigenvalue decomposition of the matrix $D$ which is discussed in the next section. These nodal coordinates are substituted into Eq. (14) to define force density vector $q$ by the single value decomposition of the equilibrium matrix $A$ which is also presented in the next section. The force density matrix $D$ is then updated by Eq. (8). The process is iteratively calculated for searching a set of nodal coordinates $[x \ y \ z]$ and force density vector $q$ until the rank deficiencies of Eqs. (17) and (20) are satisfied, which forces Eqs. (12) and (14) become true. In this context, at least one state of self-stress can be created, $s \geq 1$. In this study, based on required rank deficiencies from Eqs. (17) and (20) the form-finding process is stopped as

$$n_D^b = d + 1,$$  \hspace{1cm} (22a)

$$n_A^b = 1,$$  \hspace{1cm} (22b)

where $n_D^b$ and $n_A^b$ are minimum required rank deficiencies of the force density and equilibrium matrices, respectively.

4.1. Eigenvalue decomposition of force density matrix

The square symmetric force density matrix $D$ can be factorized as follows by using the eigenvalue decomposition \cite{20}:

$$D = \Phi \Lambda \Phi^T,$$  \hspace{1cm} (23)

where $\Phi (\in \mathbb{R}^{n \times n})$ is the orthogonal matrix $(\Phi \Phi^T - I$, in which $I \in \mathbb{R}^{n \times n}$ is the unit matrix) whose $i$th column is the eigenvector basis $\phi_i (\in \mathbb{R}^n)$ of $D$. $\Lambda (\in \mathbb{R}^{n \times n})$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, i.e., $\lambda_i = \phi_i$. The eigenvector $\phi_i$ of $\Phi$ corresponds to eigenvalue $\lambda_i$ of $\Lambda$. The eigenvalues are in increasing order as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$  \hspace{1cm} (24)

It is clear that the number of zero eigenvalues of $D$ is equal to the dimension of its null space. Let $p$ be the number of zero and
negative eigenvalues of $D$. There are two cases need to be considered. The first one is $p \leq n_D$, and the other is $p > n_D$.

**Case 1.** The first $n_D$ orthonormal eigenvectors of $\Phi$ are directly taken as potential nodal coordinates

$$[x \ y \ z] \in \Phi = [\phi_1 \ \phi_2 \ \cdots \ \phi_{n_D}].$$

The algorithm then iteratively modifies the force density vector $q$ as small as possible to make the first $n_D$ eigenvalues of $B$ become null as

$$\hat{\lambda}_i = 0 \quad (i = 1, 2, \ldots, n_D).$$

All the projected lengths $L_i (i \in \mathbb{R}^{3n_D})$ of $\Phi$ along $n_D$ directions for $B$ members are computed by

$$L = C\Phi = [(C\phi_1) \ (C\phi_2) \ \cdots \ (C\phi_{n_D})]$$

(27)

to remove one vector $\phi_i$ among $n_D$ eigenvector bases of $\Phi$ if

$$C\phi_i = 0$$

(28)
or which $\phi_i$ causes

$$\text{det} |L_0 | = 0,$$

(29)

where $L_0 (i \in \mathbb{R}^d)$ which indicates the vector of lengths of $B$ members from any combination of $d$ eigenvectors among $n_D$ above eigenvector bases in $d$-dimensional space (assuming $d = 3$) is given by

$$L_0 = \sqrt{(C\phi_1)^2 + (C\phi_2)^2 + (C\phi_{n_D})^2}.$$

(30)

Eq. (28) shows $\phi_i$ is linearly dependent with the vector $I_j$, while Eq. (29) proves at least one of $B$ members in $d$-dimensional structure has a zero length. If there is no $\phi_i$ which satisfies Eqs. (28) and (29), the first three eigenvectors of $\Phi$ are chosen as nodal coordinates $[x \ y \ z]$ for 3-dimensional tensegrity structure.

Accordingly, $D$ will finally have the required rank deficiency $n_D$ without any negative eigenvalue. It implies $D$ is positive semi-definite, and any tensegrity structure falling into this case is super stable regardless of material properties and level of self-stress coefficients [17,19,25].

**Case 2.** Where $p > n_D$, the rank deficiency may be forced to be larger than requirement or enough but $D$ may not be positive semi-definite during iteration. Additionally, the proposed form-finding procedure will evaluate the tangent stiffness matrix of the tensegrity structure which is given in [24,23,26,9] as follows:

$$K_T = K_G + K_C,$$

(31)

where

$$K_C = A \text{diag}(\frac{E_d}{E_3}) A^T,$$

(32a)

$$K_G = I_3 \otimes D$$

(32b)

in which $K_C$ is the linear stiffness matrix, $K_G$ is the geometrical stiffness matrix induced by pre-stressed or self-stressed state; $e, a$ and $I \in \mathbb{R}^d$ denote the vectors of Young’s moduli, cross-sectional areas and pre-stressed lengths of $B$ members of the tensegrity structure, respectively; $I_3 (\in \mathbb{R}^{3\times3})$ and $\otimes$ are the unit matrix and tensor product, respectively. If the tangent stiffness matrix is positive definite, then the structure is stable when its rigid-body motions are constrained; i.e., the quadratic form of $K_T$ is positive with respect to any non-trivial motion $d$ as

$$d^T(K_T)d > 0$$

(33)
or

$$\text{eig}(K_T) = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{d-r} > 0]$$

(34)

where $r = d(d + 1)/2$ is the number of independent rigid-body motions. Using this criterion, stability of any pre-stressed or tensegrity structure can be controlled by checking eigenvalues of tangent stiffness matrix of the structure [24,23]. In short, the best scenario of configuration in 3-dimensional space is formed by three best candidate eigenvalues selected from the first four eigenvector bases which correspond to the first fourth smallest eigenvalues, respectively. These eigenvalues will be gradually modified to be zero by the proposed iterative form-finding algorithm. In other words, the proposed form-finding procedure has repeatedly approximated equilibrium configuration such that

$$D[x \ y \ z] \approx [0 \ 0 \ 0].$$

(35)

4.2. Single value decomposition of the equilibrium matrix

The equilibrium matrix $A$ is computed by substituting the set of approximated nodal coordinates $[x \ y \ z]$ from Eq. (35) into Eq. (15). In order to solve linear homogeneous system (Eq. (14)) the single value decomposition [20] is carried out on the equilibrium matrix $A$:

$$A = UVW^T,$$

(36)

where $U (\in \mathbb{R}^{d\times d}) = [u_1 \ u_2 \ \cdots \ u_{n_D}]$ and $W (\in \mathbb{R}^{d\times b}) = [w_1 \ w_2 \ \cdots \ w_b]$ are the orthogonal matrices. $V (\in \mathbb{R}^{b\times b})$ is a diagonal matrix with non-negative single values of $A$ in decreasing order as

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_b \geq 0.$$

(37)

As indicated in Eq. (22b), the iterative form-finding algorithm is successful in case of $n_D = 1$. Accordingly, there are also two cases for $s$ during the iterative form-finding procedure:

**Case 1.** $s = 0$, there exists no null space of $A$. In other words, the right single value ($\sigma_0$) of $A$ in $V$ is not equal to zero. It denotes that Eq. (14) has no nonzero force density vector $q$ as a solution. In this case, if the right single vector basis ($w_1$) in $W$ corresponding to smallest singular value ($\sigma_0$) in $V$ is used as the approximated $q$, the sign of $q$ may not match with that of $q^0$. Thus, all columns of $W$ employed to compute a vector $q$ that best matches $q^0$ are scanned by form-finding procedure. The procedure stops sign-finding until the sign of all components of $w_j (j = b \ - \ 1, \ 2, \ \ldots, 1)$ is identical to that of $q^0$, i.e. sign($w_j$) = sign($q^0$). That vector $w_j$ is directly taken as the approximated $q$. In doing so, the form-finding procedure defines the approximated $q$ that matches in signs with $q^0$, such that

$$Aq = 0.$$

(38)

**Case 2.** $s = 1$, it is known [27] that the bases of vector spaces of force densities and mechanisms of any tensegrity structure are calculated from the null spaces of the equilibrium matrix. In this case, the matrices $U$ and $W$ from Eq. (36) can be expressed, respectively, as

$$U = [u_1 \ u_2 \ \cdots \ u_{n_D} | m_1 \ \cdots \ m_{d-n-D}];$$

(39a)

$$W = [w_1 \ w_2 \ \cdots \ w_{n_D} | 0];$$

(39b)

where the vectors $m (\in \mathbb{R}^{d-n-D})$ denote the $mi = dn - r_i$ infinitesimal mechanisms; and the vector $q_1 (\in \mathbb{R}^b)$ matching in signs
with \( \mathbf{q}^i \) is indeed the single state of self-stress which satisfies the homogeneous Eq. \((14)\).

In summary, the eigenvalue decomposition of force density matrix \( \mathbf{D} \) and the single value decomposition of the equilibrium matrix \( \mathbf{A} \) are performed iteratively to find the feasible sets of nodal coordinates \([x, y, z]\) and force density vector \( \mathbf{q} \) which satisfy the minimum required rank deficiencies of the force density and equilibrium matrices as presented in Eq. \((22)\), respectively.

Let the coordinates of node \( i \) be denoted as \( \mathbf{p}_i = [x_i, y_i, z_i] \in \mathbb{R}^3 \). It should be noted that since the tensegrity structure without fixed nodes is a free body in the space, the force density vector \( \mathbf{q} \) does not change under affine transformation \([28, 29, 8]\) which transforms nodes is a free body in the space, the force density vector \( \mathbf{q} \) can be simultaneously defined by proposed form-finding procedure following algorithm.

**Algorithm**

- **Step 1**: Define \( \mathbf{C} \) by Eq. \((1)\) for the given topology of tensegrity structure.
- **Step 2**: Specify the type of each member to generate initial force density vector \( \mathbf{q}^0 \) by Eq. \((21)\). Set \( i = 0 \).
- **Step 3**: Calculate \( \mathbf{D}^i \) using Eq. \((8)\).
- **Step 4**: Carry out Eq. \((23)\) to define \([x, y, z]^i\) through Eq. \((35)\).
- **Step 5**: Determine \( \mathbf{A}^i \) by Eq. \((15)\).
- **Step 6**: Perform Eq. \((36)\) to define \( \mathbf{q}^{i+1} \) through Eq. \((38)\).
- **Step 7**: Define \( \mathbf{D}^{i+1} \) with \( \mathbf{q}^{i+1} \) by Eq. \((8)\). If Eq. \((22)\) is satisfied, the solutions exist. Otherwise, set \( i = i + 1 \) and return to Step 4.
- **Step 8**: If \( \mathbf{D} \) is positive semi-definite, go to Step 10. Otherwise, specify material property, force density coefficient for each member based on force density vector found.
- **Step 9**: If Eq. \((34)\) is fulfilled, go to Step 10. Otherwise, set \( i = i + 1 \) and return to Step 4.
- **Step 10**: The process is terminated until Eq. \((43)\) has been checked. The final coordinates and force density vector are the solutions. Otherwise, set \( i = i + 1 \) and return to Step 4.

**5. Numerical examples**

Numerical examples are presented for several tensegrity structures using Matlab Version 7.4 (R2007a) \([31]\). Based on algorithm
developed, both nodal coordinates and force density vector are simultaneously defined with limited information of nodal connectivity and the type of each member. A hexagon, expandable octahedron and truncated icosahedron belong to Class 1 tensegrity structure, while the remainings are of Class 2.

5.1. Two-dimensional tensegrity structures

5.1.1. Hexagon

The initial topology of the hexagonal tensegrity structure comprising three struts and six cables (Fig. 2) which was studied by Tibert and Pellegrino [14] and Estrada et al. [10] is herein used for verification purpose. Initial nodal coordinates, member lengths and force density coefficients are unknown in advance. The only information is the incidence matrix $C$ and the type of each member which is employed to automatically assign the initial force density vector by proposed form-finding procedure as

$$q^0 = \begin{bmatrix} q_1 - q_6 = 1, q_7 - q_9 = -1 \end{bmatrix}^T.$$  \hspace{1cm} (44)

The obtained force density vector normalized with respect to the force density coefficient of the cable 1, as presented in Table 2, agrees well with those of Tibert and Pellegrino [14] and Estrada et al. [10]. The associated stable configuration of the structure is plotted in Fig. 3. The form-finding procedure converges in only one iteration with the design error $\epsilon = 8.028 \times 10^{-16}$. The structure obtained has only one self-stress state $(s = 1)$ and one infinitesimal mechanism $(m = 1)$ when its three rigid-body motions are constrained indicating it is statically and kinematically indeterminate [22]. The force density matrix $D$ is positive semi-definite, and the structure is certainly super stable regardless of materials and

| Table 3 |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| Eigenvalues of the tangent stiffness matrix of the 2-D four-strut tensegrity structure. |
| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_6$ | $\lambda_7$ |
| $e_q = 10$ | 11.2930 | 11.2930 | 14.1421 | 20.0000 | 34.1421 | 43.1334 | 43.1334 |

| Table 4 |
|------------------|------------------|------------------|------------------|
| The force density coefficients of the expandable octahedron tensegrity structure. |
| Force density coefficients | Tibert and Pellegrino [14] | Estrada et al. [10] | Present |
| $q_1 - q_{24}$ | 1.0 | 1.0 | 1.0 |
| $q_{23} - q_{40}$ | -1.5 | -1.5 | -1.5 |

Fig. 6. An expandable octahedron tensegrity structure.

Fig. 7. The obtained geometry of the expandable octahedron tensegrity structure. (a) Top view, (b) perspective view.

Fig. 8. The convergence of the proposed iterative algorithm for the expandable octahedron.
prestress levels [17–19,25]. Consequently, the proposed form-finding procedure with limited information about the incidence matrix and element prototype is indeed capable of finding a self-equilibrium stable tensegrity structure by imposing the two necessary rank deficiency conditions.

5.1.2. Four-strut tensegri

As a next example, consider a two-dimensional four-strut tensegrity structure comprising four struts and four cables (Fig. 4). Similarly, the only information is the incidence matrix C and the type of each member which is employed to automatically assign the initial force density vector by proposed form-finding procedure as

\[ q^0 = \{q_1 - q_4 = 1, q_5 - q_8 = -1\}^T. \]  

(45)

The obtained force density vector normalized with respect to the force density coefficient of the cable 1 is as follows:

\[ q = \{q_1 - q_4 = 1, q_5 - q_8 = -2\}^T. \]  

(46)

The associated configuration of the structure whose nodal coordinates are obtained and presented in Fig. 5. The form-finding procedure converges in only one iteration with the design error \( \epsilon = 2.220 \times 10^{-15} \). The structure achieved has only one self-stress state \( (s = 1) \) and no infinitesimal mechanism \( (m = 0) \) after constraining its three rigid-body motions indicating it is statically indeterminate and kinematically determinate [22].

Contrary to all the other approaches, the proposed form-finding does not require the force density matrix positive semi-definite. Particularly in this problem, the force density matrix \( D \) is negative semi-definite (possessing one negative eigenvalue) implying that the structure is not super stable. Accordingly, tangent stiffness of the structures has been investigated using Eq. (34). For simplicity, all members are assumed to have the same axial stiffness \( c_{11} = 10 \). The force density coefficients of the members are particularly set as Eq. (46). The nonzero eigenvalues of \( K_T \) after neglecting first three zero ones corresponding to three rigid-body motions are listed in Table 3 which shows the smallest is 11.2930; i.e., the structure is definitely stable.

5.2. Three-dimensional tensegrity structures

5.2.1. Expandable octahedron

The tensegrity structure consisting of 6 struts and 24 cables (Fig. 6) was investigated first by Tibert and Pellegrino [14] and then

![Fig. 9. The obtained geometry of the truncated icosahedron tensegrity structure. (a) Top view, (b) perspective view.](image-url)

![Fig. 10. The convergence of the proposed iterative algorithm for the truncated icosahedron.](image-url)

The force density matrix obtained is positive definite as Eq. (46). The nonzero eigenvalues of \( Dq_{11} \) after neglecting first three zero ones corresponding to three rigid-body motions are listed in Table 3 which shows the smallest is 11.2930; i.e., the structure is definitely stable.

![Table 5](image-url)

Main specification in form-finding procedure of the expandable octahedron tensegrity structure at every iteration.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( q_1 - q_{24} )</th>
<th>( q_{25} - q_{30} )</th>
<th>( l_1/l_2 )</th>
<th>( \epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9073</td>
<td>−1.2658</td>
<td>1.6180</td>
<td>6.321 × 10^{-1}</td>
</tr>
<tr>
<td>2</td>
<td>0.8856</td>
<td>−1.3095</td>
<td>1.6325</td>
<td>1.181 × 10^{-1}</td>
</tr>
<tr>
<td>3</td>
<td>0.8811</td>
<td>−1.3179</td>
<td>1.6330</td>
<td>2.330 × 10^{-1}</td>
</tr>
<tr>
<td>4</td>
<td>0.8802</td>
<td>−1.3196</td>
<td>1.6330</td>
<td>4.700 × 10^{-1}</td>
</tr>
<tr>
<td>5</td>
<td>0.8800</td>
<td>−1.3199</td>
<td>1.6330</td>
<td>9.000 × 10^{-4}</td>
</tr>
<tr>
<td>6</td>
<td>0.8800</td>
<td>−1.3200</td>
<td>1.6330</td>
<td>2.000 × 10^{-4}</td>
</tr>
<tr>
<td>7</td>
<td>0.8800</td>
<td>−1.3200</td>
<td>1.6330</td>
<td>3.722 × 10^{-5}</td>
</tr>
<tr>
<td>8</td>
<td>0.8800</td>
<td>−1.3200</td>
<td>1.6330</td>
<td>7.443 × 10^{-6}</td>
</tr>
<tr>
<td>9</td>
<td>0.8800</td>
<td>−1.3200</td>
<td>1.6330</td>
<td>1.489 × 10^{-6}</td>
</tr>
<tr>
<td>10</td>
<td>0.8800</td>
<td>−1.3200</td>
<td>1.6330</td>
<td>2.977 × 10^{-7}</td>
</tr>
<tr>
<td>11</td>
<td>0.8800</td>
<td>−1.3200</td>
<td>1.6330</td>
<td>5.955 × 10^{-8}</td>
</tr>
<tr>
<td>12</td>
<td>0.8800</td>
<td>−1.3200</td>
<td>1.6330</td>
<td>1.191 × 10^{-8}</td>
</tr>
<tr>
<td>13</td>
<td>0.8800</td>
<td>−1.3200</td>
<td>1.6330</td>
<td>2.382 × 10^{-9}</td>
</tr>
</tbody>
</table>

![Table 6](image-url)

The force density coefficients of the truncated icosahedron tensegrity structure.

<table>
<thead>
<tr>
<th>Force density coefficients</th>
<th>Estrada et al. [10]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_1 - q_{60} )</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>( q_{61} - q_{90} )</td>
<td>0.6775</td>
<td>0.6775</td>
</tr>
<tr>
<td>( q_{91} - q_{120} )</td>
<td>−0.3285</td>
<td>−0.3285</td>
</tr>
</tbody>
</table>
by Estrada et al. [10] using symbolic analysis and numerical method, respectively. The calculated force density vector after normalizing with respect to the force density coefficient of the cable 1, as presented in Table 4, agrees well with those of Tibert and Pelligrino [14] and Estrada et al. [10]. The associated stable configuration of the structure is presented in Fig. 7. All of struts and cables have the same length, respectively. The length ratio between struts and cables is \( l_s/l_c = 1.633 \) coinciding with the one \( l_s/l_c = 1.63299 \) stated by Estrada et al. [10]. The form-finding procedure converges in 13 iterations. The design error convergence is described in Fig. 8 with \( \varepsilon = 2.382 \times 10^{-9} \).

In order to illustrate the proposed form-finding procedure, main specification at every step of iteration is presented in Table 5. As can be seen from this table, for the practical purpose it seems the proposed algorithm sufficiently converges at iteration 6, but for the demonstration of the efficiency all iterations are displayed. The structure obtained has only one self-stress state \( (s = 1) \) and one infinitesimal mechanism \( (m = 1) \) except for its six rigid-body motions. Accordingly, it belongs to statically and kinematically indeterminate structure [22]. The force density matrices \( \mathbf{D} \) is positive semi-definite which leads structure to super stable regardless of materials and prestress levels [17–19,25].

5.2.2. Truncated icosahedron

More complicated example is the truncated icosahedron tensegrity structure with 30 struts and 90 cables which was analyzed by Estrada et al. [10]. The calculated force density vector after normalizing with respect to the force density coefficient of the cable 1, as presented in Table 6, agrees well with that of Estrada et al. [10]. The associated stable configuration of the structure is plotted in Fig. 9. The design error \( \varepsilon \) shown in Fig. 10 which exposes the convergence of the proposed form-finding in 39 iterations (a little bit more effective in convergence in comparison with 45 iterations presented by [10]) is \( 7.572 \times 10^{-9} \). The obtained structure possesses one state of self-stress \( (s = 1) \) and 55 infinitesimal mechanisms \( (m = 55) \) excluding its six rigid-body motions. The force density matrix \( \mathbf{D} \) is positive semi-definite which leads structure to super stable regardless of materials and prestress levels [17–19,25]. It is clear that the introduction of single prestress stiffens all infinitesimal mechanisms to make the structure stable.

5.2.3. Octahedral cell

The system shown in Fig. 11 has five struts and eight cables. Four of five struts form a quadrangular polygon. The calculated force density vector after normalizing with respect to the force density coefficient of the cable 1 is:

\[
\mathbf{q} = \begin{bmatrix} q_1 - q_8 = 1 \, q_9 - q_{12} = -1 \, q_{13} = -2 \end{bmatrix}^T.
\]

(47)

The associated configuration of the structure whose nodal coordinates are obtained and displayed in Fig. 12. The form-finding procedure converges in only one iteration with the design error \( \varepsilon = 3.279 \times 10^{-15} \). The structure obtained has only one state of self-stress \( (s = 1) \) and no infinitesimal mechanism \( (m = 0) \) after constraining its six rigid-body motions indicating it is statically indeterminate and kinematically determinate [22].

Similar to the force density matrix \( \mathbf{D} \) of the four-strut tensegrity, that of the octahedral cell is also negative semi-definite (possessing one negative eigenvalue) implying that the structure is not super stable. Accordingly, tangent stiffness of the structures has been investigated using Eq. (34). For simplicity, all members are assumed to have the same axial stiffness \( e_a = 2 \). The force density coefficient of each member is directly set as Eq. (47). The nonzero eigenvalues of \( \mathbf{K} \) after neglecting first six zero ones corresponding to six rigid-body motions are listed in Table 7 which shows the smallest is 0.1002; i.e., the structure is mechanically stable.

5.2.4. Tensegrity dome

A three-dimensional tensegrity dome displayed in Fig. 13 has six struts and 12 cables. The calculated force density vector after normalizing with respect to the force density coefficient of the cable 1 is:

\[
\mathbf{q} = \begin{bmatrix} q_1 - q_8 = 1.0000 \, q_9 - q_{12} = 1.4574 \, q_{13} - q_{15} = -0.5000 \, q_{16} - q_{18} = -0.5426 \end{bmatrix}^T.
\]

(48)

Fig. 14 shows the obtained self-equilibrium configuration. The form-finding procedure converges in only one iteration with the design error \( \varepsilon = 1.935 \times 10^{-15} \). The structure obtained has only one
state of self-stress \( s = 1 \) and four infinitesimal mechanisms \( m = 4 \) after constraining its six rigid-body motions indicating it is statically and kinematically indeterminate [22]. The force density matrix \( D \) is positive semi-definite which leads structure to super stable regardless of materials and prestress levels [17–19,25].

6. Concluding remarks

The advanced form-finding procedure for both Classes 1 and 2 tensegrity structures has been proposed. The force density matrix is derived from the incidence matrix and initial set of force densities formed by the vector of type of member forces. The elements of this vector consist of unitary entries +1 and \( -C_0 \) for members in tension and compression, respectively. The equilibrium matrix is defined by the incidence matrix and nodal coordinates. The eigenvalue decomposition of the force density matrix and the single value decomposition of the equilibrium matrix are performed iteratively to find the range of feasible sets of nodal coordinates and force densities which satisfy the required rank deficiencies of the force density and equilibrium matrices, respectively.

Any assumption about initial nodal coordinates, symmetric properties, member lengths and the positive semi-definite condition of the force density matrix is not necessary in the proposed form-finding procedure. A rigorous definition is given for the required rank deficiencies of the force density and equilibrium matrices that lead to a stable non-degenerate \( d \)-dimensional tensegrity structure. The eigenvalues of the tangent stiffness are also evaluated for checking the stability of the tensegrity structures. In the numerical examples, a very good convergence of the proposed method has been shown for two-dimensional and three-dimensional tensegrity structures. The proposed algorithm is strongly capable of searching novel configurations with limited information of topology and the member’s type. As a natural extension of this research, form-finding with more complicated constraints awaits further attention.

Acknowledgments

This research was supported by a grant (code#06-R&D-B03) from Cutting-edge Urban Development Program funded by the
Ministry of Land, Transport and Maritime Affairs of Korean government. The authors also would like to thank the anonymous reviewers for their suggestions in improving the standard of the manuscript.

References