Advanced form-finding for cable-strut structures

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A numerical method is presented for form-finding of cable-strut structures. The topology and the types of members are the only information that is required in this form-finding process. Dummy members are used to transform the cable-strut structure with supports into self-stressed system without supports. The requirement on rank deficiencies of the force density and equilibrium matrices for the purpose of obtaining a non-degenerate d-dimensional self-stressed structure has been explicitly discussed. The spectral decomposition of the force density matrix and the singular value decomposition of the equilibrium matrix are performed iteratively to find the feasible sets of nodal coordinates and force densities which satisfy the minimum required rank deficiencies of the force density and equilibrium matrices, respectively. Based on numerical examples it is found that the proposed method is very efficient, robust and versatile in searching self-equilibrium configurations of cable-strut structures.

1. Introduction

The Cable dome structures first proposed by Geiger et al. (1986) have been developed in recent years due to their innovative forms, lightweight and deployability. They belong to a class of prestressed pin-jointed systems that cannot be stable without introducing prestresses to some members (Pellegrino, 1992). Cable domes and tensegrity structures are included in the class of cable-strut structures as special cases.

For the form-finding problem of cable nets and tensegrity structures, there have been extensive researches. As a pioneering work of form-finding, so-called force-density method was proposed by Schek (1974) for form-finding of tensile structures. Motro et al. (1986) presented the dynamic relaxation which has been reliably applied to tense structures (Barnes, 1999) and many other nonlinear problems. Vassart and Motro (1999) employed the force density method in symbolic form for searching new configurations. Most recently, Masic et al. (2005), Zhang and Ohsaki (2006), Estrada et al. (2006) developed new numerical methods using a force density formulation, and Zhang et al. (2006a) employed a refined dynamic relaxation procedure. Tibert and Pellegrino (2003) presented a review paper for the existing methods for form-finding problem of tensegrity structures. The most recent review for this problem can be found in Juan and Tur (2008).

For form-finding problem of cable domes, Kawaguchi et al. (1999) proposed a least-square problem of nodal displacements with the specified external forces. Recently, Yuan and Dong (2002) and Yuan and Dong (2003) proposed the concept of feasible integral prestress modes considering the inherent geometric symmetry of cable domes. Ohsaki and Kanno (2003) investigated the form-finding of cables domes under specified stresses by nonlinear mathematical programming problem. Deng et al. (2005) suggested problem of shape finding of cable-strut assemblies which could be incomplete with missing or slack cables during construction by using the iterative algorithm until equilibrium equations satisfied. More recently, optimum prestressing of domes with a single or with multiple integral prestress modes is also examined by Yuan et al. (2007).

In most available form-finding methods, geometrical constraints such as: symmetric properties, member directions, some initial axial forces and nodal coordinates and/or some of member lengths must be assumed known in advance. Moreover, member force density coefficients are solved in symbolic form. For example, (i) symmetric properties are employed in a group theory so as to simplify form-finding problem as presented in Masic et al. (2005); symmetric properties, member directions together with some initial axial forces and nodal coordinates (Zhang et al. (2006b)) must be specified in advance; (ii) a number of member lengths are specified at the start in a dynamic relaxation procedure and non-linear programming, as presented in Motro et al. (1986), Barnes (1999) and Tibert and Pellegrino (2003); however, these information may not always be available or easy to define at the beginning; and (iii) force density coefficients are considered as symbolic variables which cannot be applied for structures with large number of members (Vassart and Motro, 1999; Tibert and Pellegrino, 2003).
Though there is an approach (Zhang and Ohsaki, 2006) using spectral decomposition on force density matrix (note that it is called equilibrium matrix in Zhang and Ohsaki (2006)), its core idea is to directly assign zero values to the first $d+1$ smallest eigenvalues of the force density matrix in order to impose the minimum required rank deficiency on this matrix for obtaining $d$-dimensional ($d=2$ or 3) tensegrity structure. This approach needs quite a lot of iterations to obtain the target. While in the present approach, the minimum required rank deficiencies of both the force density and equilibrium matrices are achieved by repeatedly performing the spectral and singular value decompositions on these two matrices, respectively.

In this paper, a numerical method is presented for form-finding of cable-strut structures. The structures satisfying either stability (i.e., the tangent stiffness matrix is positive-definite) or super stability (i.e., the geometrical stiffness matrix is positive-definite) can be obtained by present form-finding procedure in a few of remarkable iterations, which is more efficient and versatile than other available methods only dealing with super-stable structures (e.g. Zhang and Ohsaki (2006) and Estrada et al. (2006)) that are more restrictive in the reality. The topology and the types of members, i.e. either compression or tension are the only information that is required in this form-finding process. In other words, the initial nodal coordinates are not necessary for the present form-finding. Dummy members are used to transform the cable-strut structure with supports into self-stressed system without supports. The force density matrix is derived from an incidence matrix and an initial set of force densities assigned from prototypes, while the equilibrium matrix is defined by the incidence matrix and nodal coordinates. The spectral decomposition of the force density matrix and the singular value decomposition of the equilibrium matrix are performed iteratively to find the feasible sets of nodal coordinates. If the free nodes are numbered first, then to the fixed nodes, $C_r$ can be divided into two parts as

$$C_r = [C \ C']$$

where $C$ and $C'$ are $b \times n$ and $b \times n_f$ matrices which describe the connectivities of the members to the free and fixed nodes, respectively. Let $x, y, z \in \mathbb{R}^n$ and $x_f, y_f, z_f \in \mathbb{R}^{n_f}$ denote the nodal coordinate vectors of the free and fixed nodes, respectively, in $x, y$ and $z$ directions.

For a simple two-dimensional pre-stressed cable-strut structure as shown in Fig. 1a, which consists of five members ($b=5$, four cables and one strut) and four nodes including two free nodes ($n=2$) and two fixed nodes ($n_f=2$), the connectivity matrix $C_{(5 \times 4)}$ is given in Table 1. The equilibrium equations of the free nodes in each direction of a general pin-jointed structure given by Schek (1974) can be stated as

$$C^T Q x + C'^T Q x = p_x$$
$$C^T Q y + C'^T Q y = p_y$$
$$C^T Q z + C'^T Q z = p_z$$

where $p_x, p_y, p_z \in \mathbb{R}^b$ are the vectors of external loads applied at the free nodes in $x, y$ and $z$ directions, respectively. The symbol $(.)^T$, denotes the transpose of a matrix or vector. And $Q_0 \in \mathbb{R}^{b \times b}$ is diagonal square matrix, calculated by

$$Q_0 = \text{diag}(q)$$

where $q \in \mathbb{R}^n$ suggested in Schek (1974) is the force density vector, defined by

$$q = (q_1, q_2, \ldots, q_b)^T$$

in which each component of this vector is the force $f_i$ to length $l_i$ ratio $q_i = f_i/l_i$ ($k = 1, 2, \ldots, b$) known as force density or self-stressed coefficient in Vassart and Motro (1999). Without external loading, Eq. (3) can be rewritten neglecting the self-weight of the structure as

$$Dx = -D_x x_f$$
$$Dy = -D_y y_f$$
$$Dz = -D_z z_f$$

where matrices $D \in \mathbb{R}^{b \times b}$ and $D_f \in \mathbb{R}^{n_f \times n_f}$ are, respectively, given by

![Fig. 1](image-url)
of cable nets where all members are in tension, i.e. case of pre-stressed cable-strut systems where the matrix system does not require any fixed node, and the self-stressed system can be considered as free, forming a rigid body free in space (Vassart and Motro, 1999; Motro, 2003). In this context, Eqs. (7b) and (8b) vanish, and Eq. (6) becomes

\[ D_x = 0 \]  
\[ D_y = 0 \]  
\[ D_z = 0 \]  

where \( D \) known as force density matrix (Estrella et al., 2006; Tibert and Pellegrino, 2003) or stress matrix (Connelly, 1982, 1995, 1999) can be written directly by Connelly and Terrell (1995, 1999) and Vassart and Motro (1999) without using Eq. (7) or (8) as follows:

\[
\begin{align*}
D_{ij} &= \begin{cases} 
-q_k & \text{if nodes } i \text{ and } j \text{ are connected by member } k \\
0 & \text{otherwise}
\end{cases} \\
&= \left\{ \sum_{k=1}^{n} q_k \right\} 
\end{align*}
\]

in which \( \Omega \) denotes the set of members connected to node \( i \). For example, for the two-dimensional self-stressed structure in Fig. 1b, \( D \) can be written explicitly from Eq. (10) as

\[
D = \begin{bmatrix}
q_1 + q_2 + q_3 & -q_2 & -q_1 & -q_2 \\
-q_5 & q_3 + q_4 + q_5 & -q_4 & -q_3 \\
-q_1 & -q_4 & q_1 + q_4 + q_6 & -q_6 \\
-q_2 & -q_3 & -q_6 & q_2 + q_3 + q_6
\end{bmatrix}
\]

From Eq. (10), it is obvious that \( D \) is always square, symmetric and singular with a nullity of at least 1 since the sum of all components in any row or column is zero for any self-stressed structure. Different from the matrix \( D \) of the cable net, which is always positive-definite (Schek, 1974), \( D \) of the self-stressed structure is semi-definite due to the existence of struts as compression members, with \( q_k \) < 0. Consequently, it cannot be invertible. For simplicity, Eq. (9) can be reorganized as

\[
D \begin{bmatrix} \mathbf{x} \ \mathbf{y} \ \mathbf{z} \end{bmatrix} = [0 \ 0 \ 0]
\]

On the other hand, by substituting Eq. (8) into Eq. (9) the equilibrium equations of the self-stressed structure can be expressed as

\[
\begin{align*}
C^t \text{diag}(q)Cx &= 0 \\
C^t \text{diag}(q)Cy &= 0 \\
C^t \text{diag}(q)Cz &= 0
\end{align*}
\]

Eq. (13) can be reorganized as

\[
Aq = 0
\]

where \( A \in \mathbb{R}^{m \times n} \) is known as the equilibrium matrix in Motro (2003), defined by

\[
A = \begin{bmatrix} 
C^t \text{diag}(Cx) \\
C^t \text{diag}(Cy) \\
C^t \text{diag}(Cz)
\end{bmatrix}
\]

Eq. (12) presents the relation between force densities and nodal coordinates, while Eq. (14) shows the relation between projected lengths in \( x \), \( y \) and \( z \) directions, respectively and force densities. Both Eqs. (12) and (14) are linear homogeneous systems of self-equilibrium equations with respect to nodal coordinates and force densities, respectively.

3. Requirement on rank deficiency conditions

Let \( \mathbf{q} \) be the vector of force density and \( C \) be the incidence matrix of a \( d \)-dimensional self-stressed structure in self-equilibrium.

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<tr>
<th>Table 1</th>
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<td><strong>Member/Node</strong></td>
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<td>(6), dummy member</td>
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To solve Eq. (6) with respect to the unknown coordinates \( \mathbf{x} \), \( \mathbf{y} \) and \( \mathbf{z} \) of the free nodes, the coordinates \( \mathbf{x}_k \), \( \mathbf{y}_k \) and \( \mathbf{z}_k \) of the fixed nodes and the force density vector \( \mathbf{q} \) are known. In other words, both matrices \( D \) and \( D_f \) are constant. There are two cases related to matrix \( D \). (i) Case 1: \( D \) is non-singular, which is the case of cable nets where all members are in tension, i.e. \( q_k > 0 \) (\( k = 1, 2, \ldots, b \)); a unique solution of Eq. (6) can be easily obtained; note that only one parameter which is coordinates of the free nodes can be determined. And (ii) case 2: \( D \) is singular, which is the case of pre-stressed cable-strut systems where the matrix \( D \) is always in rank deficiency, due to the presence of struts as compression members, with \( q_k \leq 0 \). Hence, it cannot be inverted. In order to perform the advanced form-finding for the cable-strut structures in which two sets of parameter (i.e., nodal coordinates, while Eq. (14) shows the relation between projected lengths in \( x \), \( y \) and \( z \) directions, respectively and force densities. Both Eqs. (12) and (14) are linear homogeneous systems of self-equilibrium equations with respect to nodal coordinates and force densities, respectively.

3. Requirement on rank deficiency conditions

Let \( \mathbf{q} \) be the vector of force density and \( C \) be the incidence matrix of a \( d \)-dimensional self-stressed structure in self-equilibrium.
It is well known that the set of all solutions to the homogeneous system of Eq. (12) is the null space of \( \mathbf{D} \). The dimension of this null space or rank deficiency of \( \mathbf{D} \) is defined as
\[
\mathbf{n}_0 = n - r_{\mathbf{D}} \tag{16}
\]
where \( r_{\mathbf{D}} = \text{rank} (\mathbf{D}) \). It is obvious that vector \( \mathbf{1}_1 = \{1, 1, \ldots, 1\}^T \in \mathbb{R}^{n-1} \) is a solution of Eq. (12) since the sum of the elements of a row or a column of \( \mathbf{D} \) is always equal to zero. The most important rank deficiency condition related to semi-definite matrix \( \mathbf{D} \) of Eq. (12) is defined by
\[
\mathbf{n}_0 \geq d + 1 \tag{17}
\]
This condition forces Eq. (12) to yield at least \( d \) useful particular solutions (Meyer, 2000) which exclude the above vector \( \mathbf{1}_1 \) due to degenerating geometry of self-stressed structure (Tibert and Pellegrino, 2003; Zhang and Ohsaki, 2006). These \( d \) particular solutions form a vector space basis for generating a \( d \)-dimensional self-stressed structure. Therefore, the minimum rank deficiency or nullity of \( \mathbf{D} \) must be \((d + 1)\) for configuration of any self-stressed structure embedding into \( \mathbb{R}^3 \), which is equivalent to the maximum rank condition of \( \mathbf{D} \) proposed by Connelly (1982, 1995, 1999) and Motro (2003) as follows:
\[
\text{max}(r_{\mathbf{D}}) = n - (d + 1) \tag{18}
\]
Similarly, the set of all solutions to the homogeneous system of Eq. (14) lies in the null space of \( \mathbf{A} \). Let \( \mathbf{n}_A \) denote dimension of null space of the equilibrium matrix \( \mathbf{A} \) which is computed by
\[
\mathbf{n}_A = n - r_{\mathbf{A}} \tag{19}
\]
where \( r_{\mathbf{A}} = \text{rank} (\mathbf{A}) \). The second rank deficiency condition which ensures the existence of at least one state of self-stress can be stated as
\[
\mathbf{s} = \mathbf{n}_A \geq 1 \tag{20}
\]
where \( s \) is known as the number of independent states of self-stress, whilst the number of infinitesimal mechanisms is computed by \( m = dn - r_{\mathbf{A}} \), as presented in Calladine (1978) and Pellegrino and Calladine (1986). It is clear that Eq. (20) allows Eq. (14) to create at least one useful particular solution (Meyer, 2000).

In short, based on these two rank deficiency conditions, Eqs. (17) and (20), the proposed form-finding procedure searches for self-equilibrium configurations that permit the existence of at least one state of self-stress in the structure. It should be noted that these two are necessary but not sufficient conditions which have to be satisfied for any \( d \)-dimensional self-stressed structure to be in a self-equilibrium state (Connelly, 1982; Tibert and Pellegrino, 2003; Motro, 2003). The sufficient conditions for pre-stressed or self-stressed pin-jointed structures can be found in Murakami (2001), Ohsaki and Zhang (2006).

4. Form-finding process

The proposed form-finding procedure only needs to know the topology of structure in terms of the incidence matrix \( \mathbf{C} \), and type of each member, i.e. either cable or strut which is under tension or compression, respectively. Based on element type, the initial force density coefficients of cables (tension) are automatically assigned as \( +1 \) while those of struts (compression) as \( -1 \), respectively, as follows:
\[
\mathbf{q}^0 = \begin{pmatrix} \mathbf{q}_{\text{cables}}^0 & \mathbf{q}_{\text{struts}}^0 \end{pmatrix} = \begin{pmatrix} +1 & +1 & \ldots & +1 \\ -1 & -1 & \ldots & -1 \end{pmatrix}^T \tag{21}
\]
Subsequently, the force density matrix \( \mathbf{D} \) is calculated from \( \mathbf{q}^0 \) by Eq. (8). After that, the nodal coordinates are selected from the spectral decomposition of the matrix \( \mathbf{D} \) which is discussed in Section 4.1. These nodal coordinates are substituted into Eq. (14) to define force density vector \( \mathbf{q} \) by the singular value decomposition of the equilibrium matrix \( \mathbf{A} \) which is presented in Section 4.2. The force density matrix \( \mathbf{D} \) is then updated by Eq. (8). The process is iteratively calculated for searching a set of nodal coordinates \( \mathbf{[x \ y \ z]} \) and force density vector \( \mathbf{q} \) until the rank deficiencies of Eqs. (17) and (20) are satisfied, which forces Eqs. (12) and (14) become true. In this context, at least one state of self-stress can be created, \( s \geq 1 \). In this study, based on required rank deficiencies from Eqs. (17) and (20) the form-finding process is stopped as
\[
\mathbf{n}_B = d + 1 \tag{22a}
\]
\[
\mathbf{n}_A = 1 \tag{22b}
\]
where \( \mathbf{n}_B \) and \( \mathbf{n}_A \) are minimum required rank deficiencies of the force density and equilibrium matrices, respectively.

4.1. Spectral decomposition of force density matrix

The square symmetric force density matrix \( \mathbf{D} \) can be factorized as follows by using the spectral decomposition (Meyer, 2000):
\[
\mathbf{D} = \Phi \Lambda \Phi^T \tag{23}
\]
where \( \Phi \in \mathbb{R}^{n \times n} \) is the orthogonal matrix \( (\Phi \Phi^T = \mathbf{I}_n \) in which \( \mathbf{I}_n \in \mathbb{R}^{n \times n} \) is the unit matrix\) whose \( \theta^\text{th} \) column is the eigenvector basis \( \phi_i \in \mathbb{R}^n \) of \( \mathbf{D} \). \( \Lambda \in \mathbb{R}^{n \times n} \) is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, i.e., \( \lambda_i = \lambda_i \). The eigenvector \( \phi_i \) of \( \Phi \) corresponds to eigenvalue \( \lambda_i \) of \( \Lambda \). The eigenvalues are in increasing order as
\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \tag{24}
\]
It is clear that the number of zero eigenvalues of \( \mathbf{D} \) is equal to the dimension of its null space. Let \( p \) be the number of zero and negative eigenvalues of \( \mathbf{D} \). There are two cases need to be considered. The first one is \( p \leq \mathbf{n}_B \), and the other is \( p > \mathbf{n}_B \).

For case 1, the first \( \mathbf{n}_B \) orthonormal eigenvectors of \( \Phi \) are directly taken as potential nodal coordinates
\[
\mathbf{x} = [x, y, z] \in \overline{\mathbf{D}} = \{\phi_1, \phi_2, \ldots, \phi_{\mathbf{n}_B}\} \tag{25}
\]
The force density vector \( \mathbf{q} \) which is repeatedly approximated from Eq. (36) is in fact the least-square solution of the linear homogeneous system equation (14) solved by singular value decomposition technique of the equilibrium matrix \( \mathbf{A} \) as presented in Section 4.2. In other words, the algorithm iteratively modifies the force density vector \( \mathbf{q} \) as small as possible to make the first \( \mathbf{n}_B \) eigenvalues of \( \mathbf{D} \) become null as
\[
\lambda_i = 0 \quad (i = 1, 2, \ldots, \mathbf{n}_B) \tag{26}
\]
While the approach suggested in Zhang and Ohsaki (2006) is based on directly assigning zero values to the first \( \mathbf{n}_B \) eigenvalues of \( \mathbf{D} \). All the projected lengths \( \mathbf{L} \in \mathbb{R}^{\mathbf{n}_B \times n} \) of \( \overline{\mathbf{D}} \) along \( \mathbf{n}_B \) directions for \( b \) members are computed by
\[
\mathbf{L} = \mathbf{C} \overline{\mathbf{D}} = \{C \phi_1, C \phi_2, \ldots, C \phi_{\mathbf{n}_B}\} \tag{27}
\]
to remove one vector \( \phi_i \) among \( \mathbf{n}_B \) eigenvector bases of \( \overline{\mathbf{D}} \) if
\[
C \phi_i = \mathbf{0} \tag{28}
\]
or which \( \phi_i \) causes a zero length to any member among \( b \) members of the structure whose lengths are defined by
\[
L_i = \sqrt{(l_{\text{g}}^2 + l_{\text{g}}^2)^2 + (l_{\text{g}}^2)^2} \tag{29}
\]
where \( l_{\text{g}} = 0 \) (\( k = 1, 2, \ldots, b \); and assuming \( d = 3 \)) indicates the vector of lengths of \( b \) members from any combination of \( d \) singular vectors among \( \mathbf{n}_B \) above singular vector bases in \( d \)-dimensional space; and \( l_{\text{g}} = 0 \) and \( l_{\text{g}} = 1 \) denote the
coordinate difference vectors of the b members in x, y and z directions, respectively, which are calculated from

\[
\begin{align*}
\mathbf{l}^x &= \mathbf{C}\phi_x \\
\mathbf{l}^y &= \mathbf{C}\phi_y \\
\mathbf{l}^z &= \mathbf{C}\phi_z
\end{align*}
\]

Eq. (28) shows \( \phi \) is linearly dependent with the vector \( \mathbf{l} \), while Eq. (29) is very useful in checking whether there exists any member with zero length among \( b \) members of \( d \)-dimensional structure. If there is no \( \phi \), which satisfies Eq. (28) or causes a zero length to any member of the structure, the first three eigenvectors of \( \mathbf{T} \) are chosen as nodal coordinates \( [x, y, z] \) for three-dimensional self-stressed structure.

Accordingly, \( \mathbf{D} \) will finally have the required rank deficiency \( n_p \) without any negative eigenvalue. It implies \( \mathbf{D} \) is positive semi-definite; and any self-stressed structure falling into this case is super-stable regardless of material properties and level of self-stress (Connelly, 1982, 1995, 1999, 2005).

For Case 2 where \( p > n_p \), which is not considered in Zhang and Ohsaki (2006). The rank deficiency may be forced to be larger than requirement or enough but \( \mathbf{D} \) may not be positive semi-definite during iteration. Additionally, the proposed form-finding procedure will evaluate the tangent stiffness matrix of the pre-stressed cable-strut structure which is given in Murakami (2001) and Guest (2006), as follows:

\[
\mathbf{K}_T = \mathbf{K}_E + \mathbf{K}_C
\]

where

\[
\begin{align*}
\mathbf{K}_C &= \mathbf{A} \text{diag}(a_k) \mathbf{A}^T \\
\mathbf{K}_C &= \mathbf{I}_I \otimes \mathbf{D}
\end{align*}
\]

in which \( \mathbf{K}_T \) is the linear stiffness matrix, \( \mathbf{K}_C \) is the geometrical stiffness matrix induced by pre-stressed or self-stressed state; \( a_k (\forall a_k) \) and \( l_k (\forall l_k) \in \mathbb{R}^b \) denote the vectors of Young’s moduli, cross-sectional areas and pre-stressed lengths of \( b \) members of the pre-stressed cable-strut structure, respectively; \( \mathbf{I}_I \in \mathbb{R}^{bb} \) and \( \otimes \) are the unit matrix and tensor product, respectively. If the tangent stiffness matrix is positive-definite, then the structure is stable when its rigid-body motions are constrained. Using this criterion, stability of any pre-stressed or self-stressed state; and the singular value decomposition of the equilibrium matrix \( \mathbf{A} \):

\[
\mathbf{A} = \mathbf{U} \mathbf{W} \mathbf{V}^T
\]

where \( \mathbf{U} \subseteq \mathbb{R}^{bd} \), \( \mathbf{W} \subseteq \mathbb{R}^{bd} \), \( \mathbf{V} \subseteq \mathbb{R}^{bd} \) are the orthogonal matrices. \( \mathbf{V} \subseteq \mathbb{R}^{bd} \) is a diagonal matrix with non-negative singular values of \( \mathbf{A} \) in decreasing order as

\[
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_b \geq 0
\]

As indicated in Eq. (22b), the iterative form-finding algorithm is successful in case of \( n_p = 1 \). Accordingly, there are also two cases for \( s \) during the iterative form-finding procedure:

Case 1: \( s = 0 \), there exists no null space of \( \mathbf{A} \). That is, the structure is not in self-equilibrium with its current approximated nodal coordinates, which is the usual case of the structure generated from the incident matrix \( \mathbf{C} \) and the initial assigned force density vector \( \mathbf{q}^0 \). In particular, the right single value \( \sigma_s \) of \( \mathbf{A} \) in \( \mathbf{V} \) is not equal to zero. It denotes that Eq. (14) has no non-zero force density vector \( \mathbf{q} \) as a solution. In this case, if the right single vector basis \( \mathbf{w}_s \) in \( \mathbf{W} \) corresponding to smallest singular value \( \sigma_s \) in \( \mathbf{V} \) is used as the approximated \( \mathbf{q} \), the sign of \( \mathbf{q} \) may not match with that of \( \mathbf{q}^0 \). Thus, all columns of \( \mathbf{W} \) employed to compute a vector \( \mathbf{q} \) that best matches \( \mathbf{q}^0 \) are scanned by form-finding procedure. The procedure stops sign-finding until the sign of all components of \( \mathbf{w}_s \) is identical to that of \( \mathbf{q}^0 \).

Case 2: \( s = 1 \), it is known (Pellegrino, 1993) that the bases of vector spaces of force densities and mechanisms of any self-stressed structure are calculated from the null spaces of the equilibrium matrix. In this case, the matrices \( \mathbf{U} \) and \( \mathbf{W} \) from Eq. (34) can be expressed, respectively, as

\[
\begin{align*}
\mathbf{U} &= [u_1, u_2, \ldots, u_b] [m_1, m_2, \ldots, m_{d-r_A}]
\end{align*}
\]

\[
\begin{align*}
\mathbf{W} &= [w_1, w_2, \ldots, w_b] [q_1]
\end{align*}
\]

where the vectors \( \mathbf{m} \in \mathbb{R}^d \) denote the \( m = d - r_A \) infinitesimal mechanisms; and the vector \( \mathbf{q}_1 \in \mathbb{R}^b \) matching in sign with \( \mathbf{q}^0 \) is indeed the single state of self-stress which satisfies the homogeneous Eq. (14).

In summary, the spectral decomposition of force density matrix \( \mathbf{D} \) and the singular value decomposition of the equilibrium matrix \( \mathbf{A} \) are performed iteratively to find the feasible set of nodal coordinates \( [x, y, z] \) and force density vector \( \mathbf{q} \) by selecting the appropriate singular vector bases in each decomposition as the least-square solutions until the minimum required rank deficiencies of these two matrices are satisfied, respectively, as presented in Eq. (22). In fact, it is Eq. (36) that forces the structure with given topology and types of members to be in self-equilibrium with at least one state of self-stress. The final nodal coordinates \( [x, y, z] \) and force density vector \( \mathbf{q} \) are not unique for the structure with given topology and types of members. Other two sets of nodal coordinates and
force densities may exist for the same topology and the same types of members.

Let the coordinates of node \(i\) be denoted as \(p_i = [x_i, y_i, z_i] \in \mathbb{R}^3\). Since the self-stressed structure without fixed nodes is a free body in the space, the force density vector \(\mathbf{q}\) does not change under affine transformation (Tarnai, 1989; Connelly and Whiteley, 1996; Masic et al., 2005) which transforms \(\mathbf{p}\) to \(\mathbf{p}'\) as follows:

\[
\mathbf{p}' = \mathbf{p} + \mathbf{T} + \mathbf{t}
\]

where \(\mathbf{T} \in \mathbb{R}^{3 \times 3}\) and \(\mathbf{t} \in \mathbb{R}^3\) represent affine transformation which is a composition of rotations, translations, dilations, and shears. It preserves collinearity and ratios of distances; i.e., all points lying on a line are transformed to points on a line, and ratios of the distances between any pairs of the points on the line are preserved (Weisstein, 1999). Hence, numerous geometries in self-equivalence are derived by Eq. (38) with the same force density vector \(\mathbf{q}\). For simplicity, in this paper, it is assumed that \(\mathbf{t} = 0\) and \(\mathbf{T}\) is considered as unit matrix.

It should be noted that from Eqs. (12) and (22a) there are \((n_x \times d)\) components of \(n_x\) independent nodes (Vassart and Motro, 1999) in the matrix of nodal coordinates \(\mathbf{x y z}\) that can be arbitrarily specified proving that the rank of the specified nodal coordinates matrix \(\mathbf{x y z}|_{n_x < d}\) must be equal to \(d\) to avoid obtaining a degenerate \(d\)-dimensional self-stressed structure; i.e. there are 3 or 4 independent nodes whose coordinates need to be specified for two- or three-dimensional structure, respectively, in order to get the unique configuration. Hence, after obtaining the final feasible force density vector \(\mathbf{q}\) by the proposed form-finding procedure, a unique non-degenerate configuration of the self-stressed structure can also be determined based on the null space of the force density matrix \(\mathbf{D}\) and the specified independent set of nodal coordinates (Zhang and Ohsaki, 2006).

Since the self-stressed structure should satisfy the self-equilibrium conditions, the vector of unbalanced forces \(\mathbf{e}_f \in \mathbb{R}^d\) defined as follows can be used for evaluating the accuracy of the results:

\[
\mathbf{e}_f = \mathbf{Aq}
\]

The Euclidean norm of \(\mathbf{e}_f\) is used to define the design error \(\epsilon\) as

\[
\epsilon = \sqrt{\mathbf{e}_f^T \mathbf{e}_f}
\]

Two sets of parameters which are nodal coordinates and force density vector can be simultaneously defined by proposed form-finding procedure through the following algorithm.

**Algorithm**

- **Step 1:** Convert cable-strut structure to self-stressed system. Define \(\mathbf{C}\) by Eq. (1).
- **Step 2:** Specify the type of each member to generate initial force density vector \(\mathbf{q}^i\) by Eq. (21). Set \(i = 0\).
- **Step 3:** Calculate \(\mathbf{D}^i\) using Eq. (8).
- **Step 4:** Carry out Eq. (23) to define \(\mathbf{x y z}^i\) through Eq. (33).
- **Step 5:** Determine \(\mathbf{A}^i\) by Eq. (15).
- **Step 6:** Perform Eq. (34) to define \(\mathbf{q}^{i+1}\) through Eq. (36).
- **Step 7:** Define \(\mathbf{D}^{i+1}\) with \(\mathbf{q}^{i+1}\) by Eq. (8). If Eq. (22) is satisfied, the solutions exist. The process is terminated until Eq. (40) has been checked. The final coordinates and force density vector are the solutions. Otherwise, set \(i = i + 1\) and return to Step 4.

It is noted that if the geometry of cable-strut structure is known in advance, then only force density vector needs to be calculated. The algorithm skips Steps 2, 3 and 4 and does not need any iteration. If \(n_x^i = 1\) (or may be greater than 1 for the case of multiple self-stress states), the structure with given geometry is in a state of self-stress, the force density vector \(\mathbf{q}\) is found by Eq. (37b). Otherwise, i.e. \(n_x^i \neq 0\), it is not in a state of self-stress, and thus no solution is found.

## 5. Numerical examples

Numerical examples are presented for several cable-strut structures using Matlab Version 7.4(R2007a) (Yang et al., 2005). Based on algorithm developed, there are two different approaches including in the proposed form-finding. The first approach is to find only the force density vector for the cable-strut structure with given geometrical configuration in terms of nodal coordinates. In the second approach, both force density vector and nodal coordinates are simultaneously defined with limited information of nodal connectivity and the type of the each member. They are called given geometry and simultaneous approaches, respectively, in this paper.

### 5.1. Two-dimensional cable-strut structures

The initial topology of a two-dimensional cable-strut structure (Fig. 2a) comprising two struts and six cables which was studied by Zhang et al. (2006b) is herein used for verification purpose in given geometry approach. The supports can be converted into free nodes by using the dummy member in order to obtain the self-stress system. After implementation of form-finding using the method proposed in Section 4, the dummy member will be removed to transform the two nodes back to the supports. By connecting the two supports with dummy member 9, the topology of the equivalent free-standing 2D self-stressed structure is described in Fig. 2b where thin, thick and dashed lines represent the cables, struts and dummy member, respectively. Note that in

![Table 3](image)

Table 3: The specified nodal coordinates of the 2D cable-strut structure.

<table>
<thead>
<tr>
<th>Node</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<td>5.00</td>
<td>5.00</td>
<td>2.00</td>
<td>0.00</td>
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<td>2.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

![Fig. 2](image)

Fig. 2. (a) A two-dimensional cable-strut structure. (b) Its equivalent free-standing 2D self-stressed structure with dummy member to remove the supports.
the approach proposed by Zhang et al. (2006b), the directions of some members are first assigned as geometrical constraints which are incorporated in the self-equilibrium equations. Then, two sets of forces and nodal coordinates of the structures are, in turn, uniquely defined from the constrained self-equilibrium equations after specifying two independent sets of forces and nodal coordinates, respectively. This is the main difference with the present method.

For given geometry approach, nodal coordinates are given in Table 3, the obtained force density vector after normalizing with respect to the force density coefficient of the cable 1, as presented in Table 4, agrees well with that of Zhang et al. (2006b).

For simultaneous approach, no nodal coordinates as well as symmetry, member lengths and force density coefficients are known in advance. The only information is the incidence matrix C and the type of each member which is employed to automatically assign the initial force density vector by proposed form-finding procedure as

$$q^0 = (q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9)^T = (0, 0, 0, 1, 1, 1, 1, -1, -1)^T$$  (41)

The obtained force density vector normalized with respect to the force density coefficient of the cable 1 is as follows:

$$q = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}^T$$

$$= \{1.0000, 1.2808, 1.0000, 1.0000, 1.2808, 1.0000, -0.5000, -0.5000, -0.5714\}^T$$  (42)

The associated stable configuration of the structure after neglecting the dummy member 9 is plotted in Fig. 3. The form-finding procedure for the simultaneous approach converges in only one iteration with the design error $\varepsilon$ defined in Eq. (40) less than $10^{-15}$. The structures obtained by both approaches have only one self-stress state ($s = 1$) and one infinitesimal mechanism ($m = 1$) when their rigid-body motions are constrained indicating they are statically indeterminate and kinematically indeterminate (Pellegrino and Calladine, 1986). The force density matrices $D$ in both approaches are positive semi-definite, and the structures are certainly superstable regardless of materials and prestress levels (Connelly, 1982, 1995, 1999, 2005). In other words, the introduction of single pre-stress stiffens the infinitesimal mechanism to make the structures stable in all but three directions. Consequently, the proposed form-finding procedure with limited information about the incidence matrix and element prototype is indeed capable of finding a self-equilibrium stable cable-strut structure by imposing the two necessary rank deficiency conditions.

5.2. Three-dimensional cable-strut structures

5.2.1. Four-strut cable-strut

The initial topology of three-dimensional cable-strut which consists of four struts and sixteen cables shown in Fig. 4a is analyzed. Its equivalent model by applying dummy members is depicted in Fig. 4b.

For given geometry approach, the nodal coordinates are given in Table 5, the force density vector is obtained as follows after normalizing with respect to the force density coefficient of the cable 1:

$$q = \{q_1 \sim q_8 = 1.0000, q_9 \sim q_{16} = 0.5000, q_{17} \sim q_{24} = -0.5000\}^T$$  (43)

For simultaneous approach, similar to example 1, the input information is the incidence matrix C and the type of each member which is used to automatically assign the initial force density vector by proposed form-finding procedure as

$$q = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}^T$$  (44)
The calculated force density vector after normalizing with respect to the force density coefficient of the cable 1 is

\[ \mathbf{q}^I = \{q_1 \sim q_{16} = 1, q_{17} \sim q_{24} = -1\}^T \]  

The associated stable configuration of the structure after removing the dummy members (21–24) is presented in Fig. 5. The form-finding procedure in the simultaneous approach converges in only one iteration with the design error \( \varepsilon \) defined in Eq. (40) less than \( 10^{-14} \). The structures obtained in both approaches have only one self-stress state \( s = 1 \) and seven infinitesimal mechanisms \( m = 7 \) except for their six rigid-body motions. So, they belong to statically indeterminate and kinematically indeterminate structures (Pellegrino and Calladine, 1986).

5.2.2. Five-strut cable-strut

Consider the three-dimensional cable-strut with five struts and 20 cables shown in Fig. 6a. Its equivalent model is depicted in Fig. 6b.

For given geometry approach, the nodal coordinates are given in Table 6, the force density vector is obtained as follows after normalizing with respect to the force density coefficient of the cable 1:

\[ \mathbf{q} = \{q_1 \sim q_9 = 1.0000, q_9 \sim q_{16} = 1.2808, q_{17} \sim q_{20} = -0.5000, q_{21} \sim q_{24} = -0.7192\}^T \]  

The specified nodal coordinates of the 3D four-strut cable-strut structure.

<table>
<thead>
<tr>
<th>Node</th>
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<th>3</th>
<th>4</th>
<th>5</th>
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</tr>
</tbody>
</table>

Fig. 5. The obtained geometry of the three-dimensional four-strut cable-strut structure: (a) top view and (b) perspective view.

Fig. 6. (a) A three-dimensional five-strut cable-strut structure. (b) Its equivalent free-standing 3D self-stressed structure with dummy members to remove the supports.
Table 6
The specified nodal coordinates of the 3D five-strut cable-strut structure.

<table>
<thead>
<tr>
<th>Node</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tr>
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<td>-2.5000</td>
<td>2.5000</td>
<td>-2.5000</td>
</tr>
</tbody>
</table>

Fig. 7. The obtained geometry of the three-dimensional five-strut cable-strut structure: (a) top view and (b) perspective view.

\[
\mathbf{q}^0 = \{q_1 \sim q_{20} = 1, \ q_{21} \sim q_{30} = -1\}^T
\]  \( (47) \)

The calculated force density vector after normalizing with respect to the force density coefficient of the cable 1 is

\[
\mathbf{q} = \{q_1 \sim q_{10} = 1.0000, \ q_{11} \sim q_{20} = 1.1206, \ q_{21} \sim q_{25} = -0.5000, \ q_{26} \sim q_{30} = -0.8794\}^T
\]  \( (48) \)

The associated stable configuration of the structure after neglecting the dummy members (26–30) is plotted in Fig. 7. The design error \( (e) \) is about \( 10^{-14} \). For both approaches, the obtained structures possess one self-stress state \( (s = 1) \) and 10 infinitesimal mechanisms \( (m = 10) \) excluding their six rigid-body motions. In this problem, the force density matrix \( \mathbf{D} \) is negative semi-definite indicating the structure is not super-stable. Accordingly, tangent stiffness of the structures has been investigated and found to be positive. It confirms that the structures are mechanically stable (Murakami, 2001; Ohsaki and Zhang, 2006).

6. Concluding remarks

The advanced form-finding procedure for cable-strut structures has been proposed. The force density matrix is derived from the incidence matrix and initial set of force densities formed by the vector of type of member forces. The elements of this vector consist of unitary entries +1 and −1 for members in tension and compression, respectively. The equilibrium matrix is defined by the incidence matrix and nodal coordinates. The spectral decomposition of the force density matrix and the singular value decomposition of the equilibrium matrix are performed iteratively to find the range of feasible sets of nodal coordinates and force densities. A rigorous definition is given for the required rank deficiencies of the force density and equilibrium matrices that lead to a stable non-degenerate \( d \)-dimensional self-stressed structure. In the numerical examples, a very good convergence of the proposed method has been shown for two-dimensional and three-dimensional cable-strut structures by using both given geometry and simultaneous approaches. The proposed algorithm is strongly capable of searching novel configurations with limited information of topology and the member’s type. However, all of member lengths cannot be directly controlled since they are not explicitly described in the formulation. As a natural extension of this research, form-finding with more complicated constraints awaits further attention.

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References


